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EXCITATION EFFICIENCY OF SURFACE WAVES OVER CORRUGATED METAL AND DOUBLY CORRUGATED METAL AND IN DIELECTRIC SLABS ON A GROUND PLANE

ALAN F. KAY

TECHNICAL RESEARCH GROUP
17 UNION SQUARE WEST
NEW YORK 3, NEW YORK

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ABSTRACT

Corrugated metal (and "doubly corrugated metal") is an anisotropic boundary which supports only a hybrid surface wave mode in directions other than parallel or perpendicular to the corrugations. The mode phase velocity is a function of direction. To determine the excitation efficiency of these modes a three dimensional analysis is performed generalizing the results of Cullen [1] and utilizing the author's previous work [2].

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The surface waves are shown to possess many of the properties of plane waves in a two dimensional anisotropic medium - notably that the energy propagation is radial and not generally normal to the wave fronts. Radial propagation, however, does not imply that the primary pattern of a feed and the resulting far pattern of the surface wave are the same. The hybrid plane wave component of the spectrum of the source which propagates in a direction ψ parallel to the ground plane with the natural surface wave phase velocity of the boundary in this direction is observed in the far field in a direction ϕ not generally equal to ψ ! Moreover the excitation efficiency is not independent of ψ. Therefore, the possibility of focussing by an anisotropic boundary remains.

In Part II Cullen's formula for surface wave excitation efficiency over a dielectric slab and ground plane is also generalized to thick slabs. The wide range of efficiencies obtainable as a function of slab thickness, dielectric constant, and source height is related to many of the difficulties experienced in constructing surface wave antennas.

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PART I

EXCITATION OF SURFACE WAVES IN FREE SPACE OVER AN INFINITE PLANE ANISOTROPIC BOUNDARY

INTRODUCTION

Cullen[1] has considered the excitation efficiency of a surface wave over an infinite plane corrugated metal boundary launched by an infinite line of magnetic current parallel to the corrugations. In this paper we consider the more general problem: of an arbitrary source over corrugated metal and other boundaries to be described.

From a three dimensional point of view corrugated metal is an anisotropic boundary. The surface waves excited on it have phase velocity dependent on the direction of propagation and many of the other properties of plane waves in a two dimensional anisotropic medium. The surface wave front created by a point source near an anisotropic boundary is, in fact, a more complicated curve than the elliptical wave front of a point source in a two dimensional anisotropic medium. The boundary anisotropy may cause focussing or defocussing of the source - a phenomenon with no counterpart in Cullen's two dimensional problem.

In practical microwave work one is sometimes confronted with a three dimensional problem in which there is a plane of symmetry both with respect to the geometry and with respect to the incident field. In this case it is common to assume that the solution to the problem in the plane of symmetry is not very different from that of the "two dimensionalized" problem which one obtains by assuming that the incident field and the geometry are the same in the plane of symmetry as in the original problem and are independent of the coordinate perpendicular to this plane. This assumption is implicit in [1], for although in the main body of the paper Cullen clearly states that he is solving a two dimensional problem, in his section on experimental verification he describes a three dimensional set up. His experimental source is a half wave long slot rather than the infinite magnetic line current assumed in his theory. The failure to distinguish between these two sources, might very well not offend the microwave intuition of anyone. However, in this paper the efficiencies of surface wave excitation are compared for the two sources and they are found to differ both because of inherent differences between two and three dimensions (this applies both to the corrugated and to the dielectric clad ground plane) and also in the case of the corrugated ground plane, because of the anisotropy. In certain cases the difference may be numerically appreciable.*

If the corrugations are parallel to the plane $x = 0$ and the z axis extends out of the grooves then in the two dimensional case the boundary is characterized by a normal impedance condition:

(1) $$\left.\frac{E_x}{H_y}\right|_{z=0} = -Z$$

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* See Figures 9a, b, c, and d.

If the thickness of the teeth of the corrugations is negligible compared to the gap between them then

(2) $$Z \simeq -i\sqrt{\frac{\mu}{\epsilon}}\tan(kd)$$

where d is the depth of the grooves and $k = \omega\sqrt{\mu\epsilon}$ is the free space propagation constant[1]. The same elementary considerations which lead to (2) show that

(3) $$\left.\frac{E_y}{H_x}\right|_{z=o} = 0.$$

Before proceeding it is useful to generalize the problem from the case of ordinary or single corrugations to the case of a "doubly corrugated surface" shown in Figure 1.

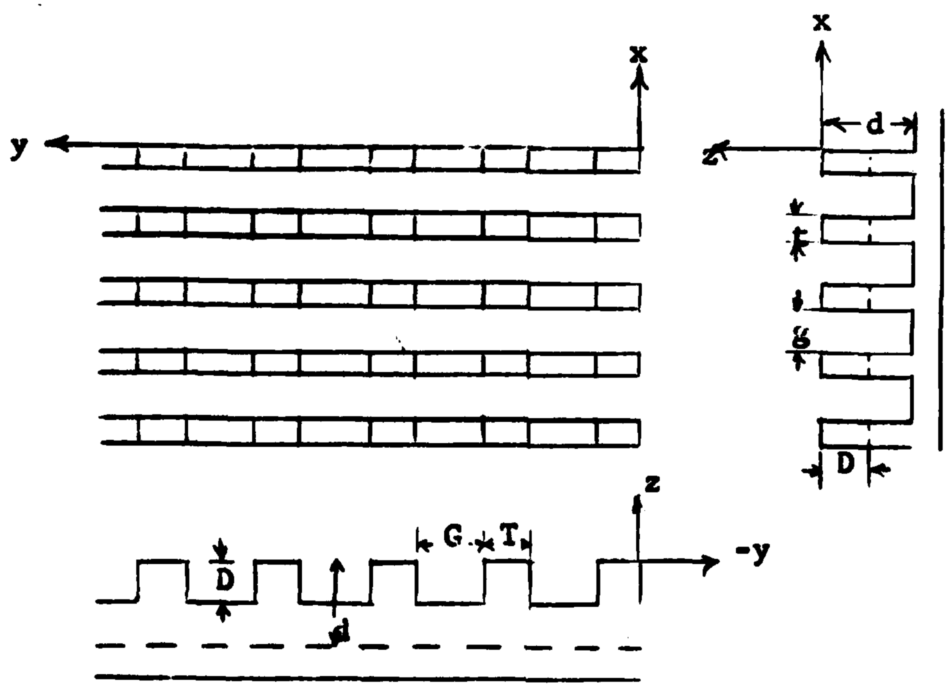


FIGURE NO. 1
DOUBLY CORRUGATED SURFACE

Such a surface may be made by milling on a flat metal plate a set of parallel grooves of depth d , width g , and tooth width t , then rotating the plate 90° and milling a second set of grooves of depth D, width G, and tooth width T. If the x direction is taken parallel to the first set and the y direction parallel to the second set, then to replace (1) - (3) we have the following boundary conditions

$$(4) \qquad \left.\frac{E_x}{H_y}\right|_{z=0} = -Z \simeq i\left(\frac{g}{t+g}\right)\sqrt{\frac{\mu}{\epsilon}}\tan(kd),$$

$$(5) \qquad \left.\frac{E_y}{H_x}\right|_{z=0} = Z_e \simeq -i\left(\frac{G}{T+G}\right)\sqrt{\frac{\mu}{\epsilon}}\tan(kD).$$

Here G, T, g, and t must all be assumed small compared to the wavelength. Obviously, if we take $D=0$ or $G=0$, we return to the case of single corrugations.

SOLUTION OF A PLANE WAVE INCIDENT ON A DOUBLY CORRUGATED BOUNDARY

The two normal impedance conditions (4) and (5) determine the scattered field when any field is incident upon the surface. This will be shown in general later. Let us first assume that a TM plane wave is incident at angle θ to the outward surface normal with the plane of incidence at angle ψ to the positive x axis. Let

$$(6) \qquad p = k\sin\theta, \qquad w = k\cos\theta = \sqrt{k^2-p^2}$$

and let ρ, ϕ, and z be the cylindrical coordinate system associated with the rectangular coordinates in the usual way. Then the incident plane wave field may be written as

$$(7) \qquad H = \frac{e^{-iwz+ip\rho\cos(\phi-\Psi)}}{w}(\sin\Psi,\ -\cos\Psi,\ 0)$$

$$\omega\epsilon E = e^{-iwz+ip\rho\cos(\phi-\Psi)}(\cos\Psi,\ \sin\Psi,\ \frac{p}{w})$$

We use here the notation (a, b, c) for the x, y, and z components respectively of a vector.

In order to satisfy the boundary conditions (4) and (5) it is necessary in general to postulate two plane waves reflected from the surface, one TE and one TM, rather than simply a TM reflected plane wave as is sufficient in the case of an isotropic boundary condition. Mathematically, this is true simply because we need in general two reflection coefficients R_1 and R_2 to satisfy the two conditions (4) and (5). Let the reflected TM wave be

$$(8) \qquad H = \frac{R_1 e^{iwz+ip\rho\cos(\phi-\Psi)}}{w}(\sin\Psi,\ -\cos\Psi,\ 0)$$

$$\omega\epsilon E = R_1 e^{iwz+ip\rho\cos(\phi-\Psi)}(-\cos\Psi,\ -\sin\Psi,\ \frac{p}{w})$$

and the reflected TE wave be

$$(9) \qquad kH = R_2 e^{iwz+ip\rho\cos(\phi-\Psi)}(\cos\Psi,\ \sin\Psi,\ -\frac{p}{w})$$

$$\frac{k}{\omega\mu}E = \frac{R_2}{w}\, e^{iwz+ip\rho\cos(\phi-\Psi)}(\sin\Psi,\ -\cos\Psi,\ 0).$$

Each of these waves satisfies Maxwell's equations and propagates outward to infinity. The total field will satisfy the boundary conditions if, as we find upon substitution in (4)-(5),

$$(10)\quad -R_1\left(\frac{1}{w} + \frac{1}{\omega\epsilon Z_o}\right)\sin\psi + \frac{R_2}{k}\left(1 + \frac{\omega\mu}{wZ_o}\right)\cos\psi = \sin\psi\left(\frac{1}{w} - \frac{1}{\omega\epsilon Z_o}\right)$$

$$R_1\left(\frac{1}{w} + \frac{1}{\omega\epsilon Z}\right)\cos\psi + \frac{R_2}{k}\left(1 + \frac{\omega\mu}{wZ}\right)\sin\psi = \cos\psi\left(\frac{1}{\omega\epsilon Z} - \frac{1}{w}\right) .$$

This pair of simultaneous linear equations may readily be solved to yield

$$(11)\quad R_1 = \frac{(\gamma + X_o)(\gamma X+1)\sin^2\psi + (\gamma + X)(\gamma X_o+1)\cos^2\psi}{(\gamma - X_o)(\gamma X+1)\sin^2\psi + (\gamma - X)(\gamma X_o+1)\cos^2\psi} = 1 + \frac{2(X_o\sin^2\psi + X\cos^2\psi + \gamma XX_o)}{\gamma^2(X_o\cos^2\psi + X\sin^2\psi) + \gamma(1-XX_o) - X\cos^2\psi - X_o\sin^2\psi}$$

$$(12)\quad R_2 = \frac{2i\sin\psi\cos\psi\,\gamma\,(X_o - X)}{(\gamma - X_o)(\gamma X+1)\sin^2\psi + (\gamma - X)(\gamma X_o+1)\cos^2\psi}$$

where we have set

$$(13)\quad \gamma = \frac{w}{ik}\ , \quad X = i\sqrt{\frac{\epsilon}{\mu}}\, Z, \quad X_o = iZ_o\sqrt{\frac{\epsilon}{\mu}} .$$

In the lossless, doubly corrugated surface case, X and X_o are real. If γ is real and positive the reflected waves are surface waves traveling in direction ψ .

We have thus solved the problem of reflection of an incident TM plane wave. In order to do the same for a

TE plane wave, we can use the duality principal on all of the preceeding equations. The duality principal states that these equations remain correct under the substitutions:

(15) $$E \rightleftarrows H$$
$$\mu \rightleftarrows -\epsilon$$
$$Z \rightleftarrows -1/Z_o$$
$$X \rightleftarrows -1/X_o$$
$$TE \rightleftarrows TM$$

if these substitutions are made wherever these symbols occur. The results give the solution for the TE plane wave. An arbitrary source can be resolved into two sources producing a continuum of TE and TM plane waves respectively. This is proved in [2]. We will next show how to find the total solution for the TM part of a general source. When this has been done, we can apply the substitutions (15) to the TM solution in order to yield the solution for the TE part of the source, and thus the total solution. Bearing in mind that there is no essential loss in generality thereby, we confine our attention exclusively to a TM incident field for the remainder of the paper. Before doing so, however, it is worth noting that the poles of R_1 and R_2 as functions of p are unchanged by the substitutions (15). This implies, as we shall see, that in general the surface waves which can be produced by TE and TM sources are the same. In any case no surface waves exist other than those corresponding to the poles of R_1 (or R_2, which has the same poles as R_1).

SURFACE WAVES

When the denominator of R_1 in (11) is zero a free solution to Maxwell's equations can exist which satisfies the boundary conditions and has many of the features usually associated with a surface wave. We shall derive these surface wave fields in a later section. Let us, for the present, however, simply examine when they can occur. The condition for a surface wave is that its vertical attenuation constant shall be γ_o, the particular value of γ which is positive and satisfies:

(16a) $$\gamma_o^2(X_o\cos^2\psi + X\sin^2\psi) + \gamma_o(1-XX_o) - X\cos^2\psi - X_o\sin^2\psi = 0$$

or

(16b) $$\tan^2\psi = f(\gamma_o)g(\gamma_o), \quad f(\gamma) = \frac{X-\gamma}{1+\gamma X}, \quad g(\gamma) = \frac{1+\gamma X_o}{\gamma - X_o}$$

(16a) and (16b) are quadratic equations for γ_o, having at most two roots. These are:

(17) $$\gamma_o = \frac{XX_o - 1 \pm \sqrt{(1+XX_o)^2 + (X-X_o)^2\sin^2 2\psi}}{2\left[X + (X_o - X)\cos^2\psi\right]} .$$

When the denominator vanishes, $\tan^2\psi = \frac{-X_o}{X}$ and

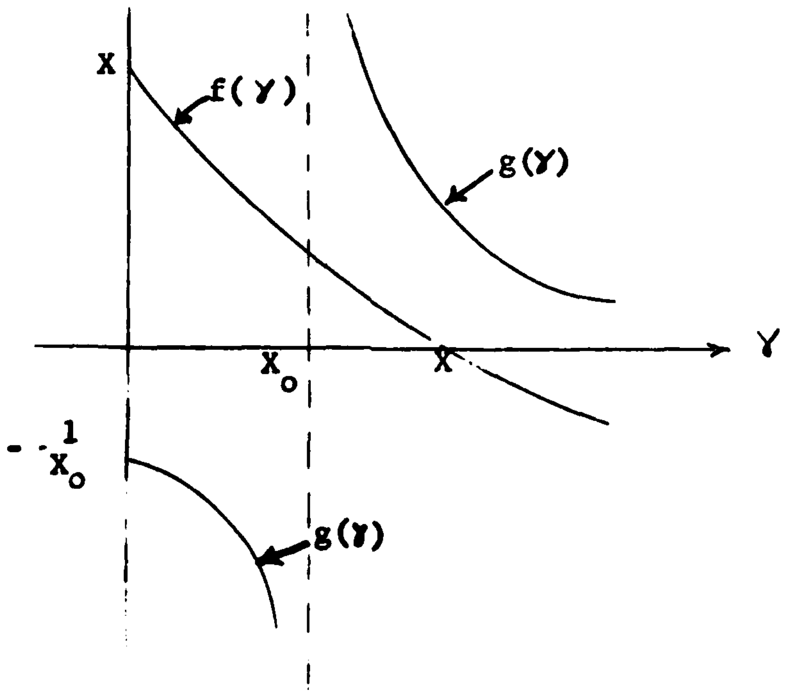
(18) $$\gamma_o = \frac{X + X_o}{(X-X_o)(1-XX_o)} = \frac{(1-\tan^2\psi)}{(1+X^2\tan^2\psi)(1+\tan^2\psi)} .$$

If X and X_o are both positive, then the only positive value of γ_o corresponds to the positive sign for the radical. If

X and X_o are both negative, then the negative sign is required for the radical. If one is positive and the other negative, either sign may be required. In order to determine the number of positive solutions of (17) and some of their properties as functions of Ψ we now consider various cases separately.

(i) Pure Inductive Case

Assume X and X_o are both positive. Without loss in generality, let $X \geq X_o$. For each Ψ, there is then one and only one γ_o satisfying (16). We may see this as follows: The left member of (16b) is always positive. The right member can only be positive if γ_o is between X_o and X. In the range $X_o \leq \gamma \leq X$, both $f(\gamma)$ and $g(\gamma)$ are positive decreasing functions. Therefore, their product is also a monotone function which decreases from infinity to 0 in this interval

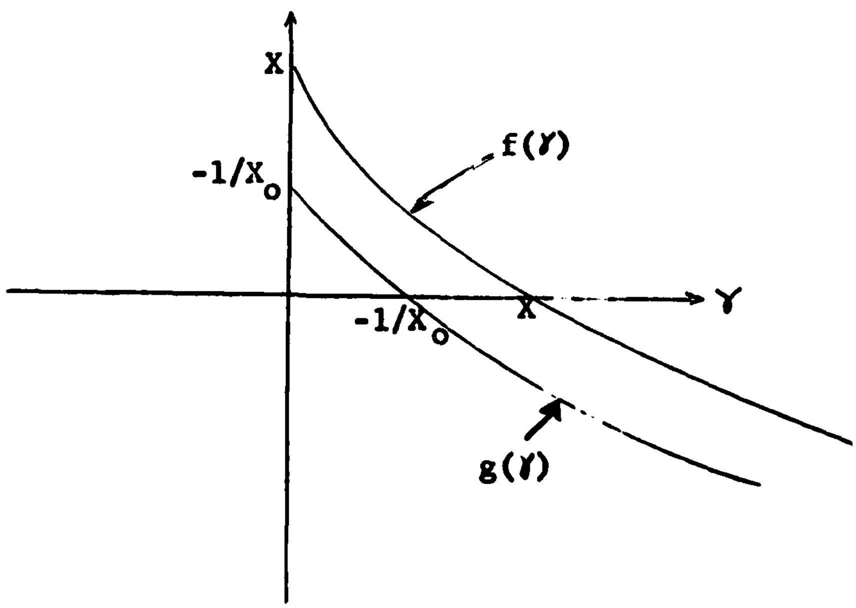


Since the left member is a monotone increasing function for $0 \leq \psi \leq \pi/2$, there can be only one solution $\gamma_o = \gamma_o(\psi)$. Furthermore, this solution equals X when $\psi = 0$ and decreases continuously, as ψ increases, reaching a value X_o when $\psi = \pi/2$. The values repeat periodically in the other quadrants - as indeed they must - for, from the symmetry of the problem, there is no distinction between any angle ψ and the related angles $-\psi$, $\pi - \psi$, or $\pi + \psi$. Therefore γ_o is a maximum when $\psi = 0$ and π and a minimum when $\psi = \pi/2$ and $3\pi/2$, and accordingly

$$(19) \qquad \gamma_o' \equiv \frac{\partial \gamma_o(\psi)}{\partial \psi} = \begin{cases} < 0 \text{ when } 0 < \psi < \pi/2 \\ 0 \text{ when } \psi = 0,\ \pi/2,\ \pi, \text{ or } 3\pi/2. \end{cases}$$

<u>(ii) Mixed Case:</u> $X > 0$, $X_o < 0$

(iia) $XX_o < -1$ or $X > -\frac{1}{X_o}$



In this case, in the interval $0 \leq \gamma \leq -1/X_o$, both $f(\gamma)$ and $g(\gamma)$ are positive decreasing functions, hence so is their product. Equation (16) has a unique root here if and only if

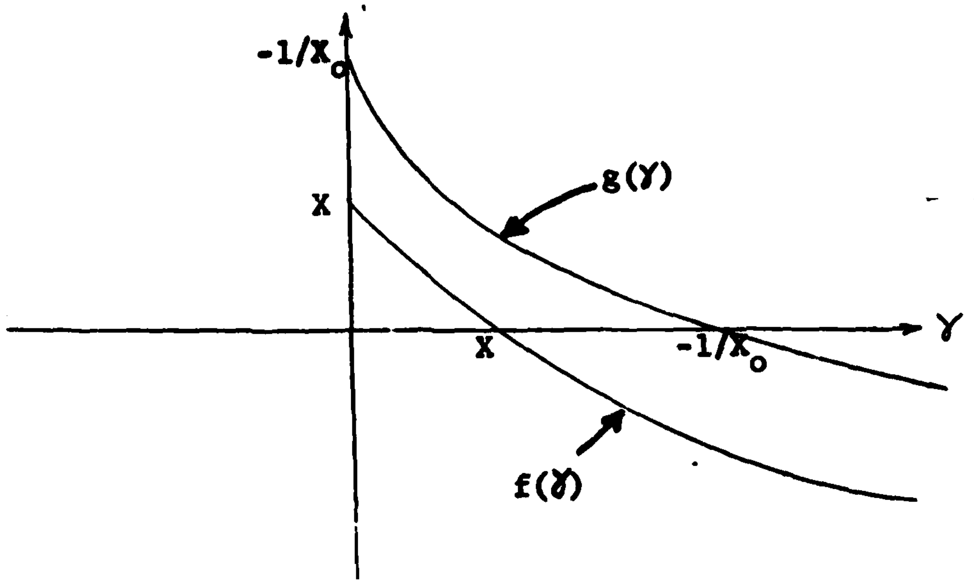
(20) $$0 \leq \tan^2 \psi \leq -\frac{X}{X_o} \quad .$$

For this root $\gamma_o' < 0$. In the interval $-\frac{1}{X_o} \leq \gamma \leq X$, $f(\gamma)$ is positive and $g(\gamma)$ is negative, so that there are no solutions of (16). If $\gamma \geq X$, both $f(\gamma)$ and $g(\gamma)$ are negative decreasing functions so that their product is a positive increasing function. Equation (16) then has a unique root in this interval if and only if

(21) $$0 \leq \tan^2 \psi \leq -\frac{X_o}{X},$$

and for this root $\gamma_o' > 0$. These facts may be summarized as follows: Let $m = \min\left\{-\frac{X_o}{X}, \; -\frac{X}{X_o}\right\}$. Then $m \leq 1$. We conclude that if $\tan^2 \psi \leq m$, there are two values of γ_o satisfying (16), one in the interval $0 \leq \gamma \leq -1/X_o$ and one satisfying $X \leq \gamma$. If $m \leq \tan^2 \psi \leq \frac{1}{m}$ there is only one value of γ_o satisfying (16) and it lies in the interval $0 \leq \gamma \leq -1/X_o$ if $m = -\frac{X_o}{X}$, or in $\gamma \geq X$ if $m = -\frac{X}{X_o}$. If $\tan^2 \psi > \frac{1}{m}$, (16) has no solutions. The root in the interval $0 \leq \gamma \leq -1/X_o$ is a decreasing function of ψ :, i.e., $\gamma_o' < 0$, and the root in the interval $X \leq \gamma$ is an increasing function of ψ : i.e., $\gamma_o' > 0$. This case is illustrated in Figure 7 with $X = 1$, $X_o = -4$.

(iib) $XX_o > -1$



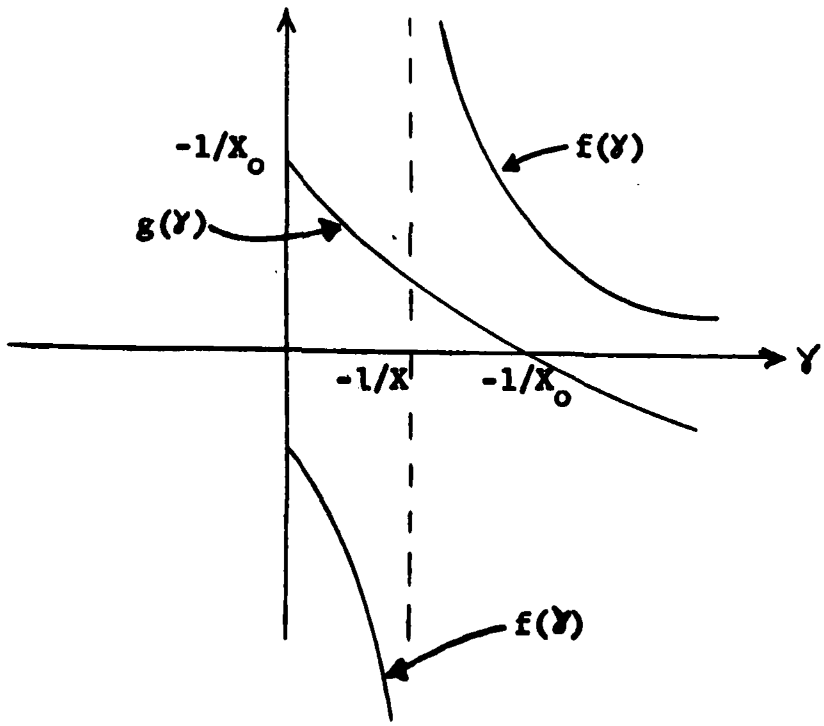
The conclusions in this case are similar to (iia).
If $\tan^2\psi \le m$, there are two solutions to (16), one in the interval $0 \le \gamma \le X$ and one in the interval $-1/X_o \le \gamma$. If $m \le \tan^2\psi \le \frac{1}{m}$, there is only one solution. It satisfies $-1/X_o \le \gamma$ if $m = -\frac{X}{X_o}$ or $0 \le \gamma \le X$ if $m = -\frac{X_o}{X}$. Both cases (ii a) and (ii b) may be summarized together if we let $M' = \max\left\{X, -\frac{1}{X_o}\right\}$, $m = \min\left\{-\frac{X_o}{X}, \frac{-X}{X_o}\right\}$ and $m' = \min\left\{X, -1/X_o\right\}$. Then if $\tan^2\psi \le m$, there are two solutions to (16), one in the interval $0 \le \gamma \le m'$ (for which $\gamma_o' < 0$) and one in the interval $M' \le \gamma$ (for which $\gamma_o' > 0$). If $m \le \tan^2\psi \le \frac{1}{m}$, there is only one solution. It satisfies $M' \le \gamma$ if $m = -\frac{X}{X_o}$ or

$0 \le \gamma \le m'$ if $m = -\frac{X_o}{X}$. If $\tan^2\psi \ge \frac{1}{m}$, there are no solutions to (16).

(iii) Mixed Case: $X_o > 0$, $X < 0$.

Here if we let $M'' = \max\left\{X_o, -\frac{1}{X}\right\}$, $m'' = \min\left\{X_o, -\frac{1}{X}\right\}$. Then if $\frac{1}{m} \le \tan^2\psi$, there are two distinct solutions to (16), one in the interval $0 \le \gamma \le m''$ (for which $\gamma_o' > 0$) and one in the interval $M'' \le \gamma$ (for which $\gamma_o' < 0$). If $m \le \tan^2\psi \le \frac{1}{m}$, there is one solution. It satisfies $M'' \le \gamma$, if $m = -\frac{X}{X_o}$, or $0 \le \gamma \le m''$, if $m = -\frac{X_o}{X}$. If $\tan^2\psi \le m$ there are no solutions.

(iv) Pure Capacitive Case: $X_o < 0$, $X < 0$



In this case, for each ψ there is a unique positive γ_0 satisfying (16). This value of γ_0 lies between $-1/X$ and $-1/X_0$. If we assume $X_0 \angle X$, then $\gamma_0' \angle 0$.

FIELD PRODUCED BY A SOURCE OVER A DOUBLY CORRUGATED BOUNDARY

Equation (38) of reference [2] gives the representation of a more or less arbitrary incident TE field as a superposition of plane and hybrid waves weighted according to the far field of the ϕ-component of E for the source in free space. Equation (125) of [2] gives a similar representation for the scattered field. In order to utilize these results here we must first employ the duality substitutions (15). We must then make a change in coordinates (in [2] positive z points into the boundary). Finally we must include the pair of reflected plane waves instead of the single plane wave reflected from an isotropic boundary. If these steps are performed correctly, we obtain the total field, incident plus scattered, for a TM source:

(22)

$$H = \frac{i}{2\pi k}\int_0^{\infty}\int_0^{2\pi} \frac{pe^{ip\rho\cos(\phi-\psi)+iwz}}{w}\, dp\, d\psi \Big\{-k\sin\psi(U_+ + R_1U_-) - R_2 w\cos\psi U_-,\ k\cos\psi(U_+ + R_1U_-) - R_2 w\sin\psi U_-,\ R_2 p U_-\Big\}.$$

$$E = \frac{i}{2\pi\omega\epsilon}\int_0^{\infty}\int_0^{2\pi} \frac{pe^{ip\rho\cos(\phi-\psi)+iwz}}{w}\, dp\, d\psi \Big\{w\cos\psi(U_+ + R_1U_-) - kR_2\sin\psi U_-,\ w\sin\psi(U_+ + R_1U_-) + kR_2\cos\psi U_-,\ -p(U_+ + R_1U_-)\Big\}.$$

The symbolism U_+ and U_- has the following meaning: the ϕ component of H, when the source is in free space, assumes the form

$$(23) \qquad F(\theta,\phi)\frac{e^{ikr}}{r} + O\left(\frac{1}{r^2}\right) \text{ as } r \longrightarrow \infty$$

where $F(\theta,\phi)$, the "far field", may be rewritten in terms of $p = k\sin\theta$ and $\psi = \phi$ as a new function $U(p,\psi) = F(\sin^{-1}p/k, \psi)$. An ambiguity exists in this definition since the inverse sine has two values: $\pi-\theta$ as well as θ. The far fields in these two directions are not generally the same, of course. (An exception in the plane $\phi = 0$ is the magnetic line current along the y axis. Here $F(\theta,0)$ is a constant). $U(p,\psi)$ is thus generally a double valued function. In (22) we have let U_+ stand for the value of $U(p,\psi)$ corresponding to directions for which θ is less than $\pi/2$ and U_-, the other value, for which θ exceeds $\pi/2$. Our problem now is to obtain both the radiation field and the surface wave field from (22). The former is the asymptotic value of these integrals for large r, where $\rho = r\sin\theta$, $z = r\cos\theta$. The latter is the asymptotic value when ρ is large and z is fixed. Faced with the same problem for the isotropic boundary case, the author required considerable mathematical analysis in reference [2] to obtain just the surface wave field rigorously. The method employed there cannot apply directly here since now the reflection coefficients have an angular dependence: R_1 and R_2 are functions of ψ.

Instead here we outline a proof which it is believed could be made rigorous though not without great effort. We present it simply as a plausible argument. We begin by writing each of the integrals of (22) as a sum of two terms, one containing the ψ integration from $-\pi/2 + \phi$ to $\pi/2 + \phi$, and the other the integration over the remaining interval from $\pi/2 + \phi$ to $3\pi/2 + \phi$. We indicate this symbolically with H as

$$(24) \qquad H = H_1 + H_2, \quad H_1 = \int_0^\infty \int_{\phi-\pi/2}^{\phi+\pi/2} , \; H_2 = \int_0^\infty \int_{\phi+\pi/2}^{\phi+3\pi/2} ,$$

since there is no need here to bore the typist with the lengthy integrands. Now we observe that $U(p,\psi) = U(-p,\psi + \pi)$, since the direction (θ,ϕ) is the same as $(-\theta,\phi+\pi)$. Thus, if in the integrand of H_2 we make the change of variable $\psi' = \psi + \pi$, $p' = -p$, we find that

$$(25) \qquad H_2 = \int_{-\infty}^{0} \int_{\phi-\pi/2}^{\phi+\pi/2}$$

so that

$$(26) \qquad H = \int_{-\infty}^{\infty} \int_{\phi-\pi/2}^{\phi + \pi/2}$$

(The details of indentation of the p integration contour about the poles of R_1 and R_2, of course, must be properly handled). In the p integration we have succeeded, therefore, in obtaining an infinite contour which may - if the integrand behaves properly at infinity - be deformed around the poles and branch cut. In the ψ integration, we have avoided the stationary phase point in the backward direction $\psi = \phi + \pi$ and are left with the only physically reasonable stationary phase point at $\psi = \phi$. An estimate of $U(p, \psi)$ for large p is required. It is not hard to show, from the formulas of [2], that

$$(27) \qquad U(p,\psi) = O(e^{\rho_0 \mathscr{I}m(p) - |\mathscr{R}e(p)|\delta}), \quad p \longrightarrow \infty$$

where ρ_0 is the radius of the smallest sphere containing the source and δ is the minimum distance of the source to the plane $z = 0$. A formula similar to (27) would be required in a rigorous proof.

RADIATION FIELD

The radiation field follows from a stationary phase evaluation of the integrals. When the observation direction is (θ,ϕ) the stationary phase point occurs when $\psi = \phi$, $p = k\sin\theta$ (and $w = +k\cos\theta$). The results may be most simply expressed in terms of the θ and ϕ components of the field as follows:

$$(28) \qquad H_\phi = F(\theta,\phi) + R_1 F(\pi-\theta,\phi)$$

$$H_\theta = -R_2 F(\pi-\theta,\phi)$$

$$E_\phi = \frac{\omega\mu}{k} R_2 F(\pi-\theta,\phi)$$

$$E_\theta = \frac{k}{\omega\epsilon} (F(\theta,\phi) + R_1 F(\pi-\theta,\phi))$$

where R_1 and R_2 are evaluated at $\gamma = -i\cos\theta$, $\psi = \phi$:

$$(29) \quad R_1 = \frac{(X\cos^2\theta - X_o)\sin^2\phi + (X_o\cos^2\theta - X)\cos^2\phi + i\cos\theta(1+XX_o)}{(i\cos\theta+X_o)(1-iX\cos\theta)\sin^2\phi + (i\cos\theta+X)(1-iX_o\cos\theta)\cos^2\phi}$$

$$= 1 + \frac{2(iXX_o\cos\theta - X_o\sin^2\phi - X\cos^2\phi)}{(X_o\cos^2\phi + X\sin^2\phi)\cos^2\theta - i\cos\theta(XX_o-1) + X\cos^2\phi+X_o\sin^2\phi}$$

$$R_2 = \frac{2\sin\phi\cos\phi\cos\theta(X-X_o)}{(X_o\cos^2\phi + X\sin^2\phi)\cos^2\theta - i\cos\theta(XX_o-1) + X\cos^2\phi + X_o\sin^2\phi}$$

The power flow between two meridians: $\phi_o < \phi < \phi_1$ is

$$(30) \quad P_r = \frac{\mathcal{R}_e}{2}\left\{\int_0^{\pi/2} \sin\theta d\theta \int_{\phi_o}^{\phi_1} d\phi(E\times\bar{H})\cdot\bar{n}\right\}$$

where $\bar{n}$ is the unit radial vector. Substitution of (28) yields

$$(31) \quad P_r = \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}} \int_0^{\pi/2} \sin\theta d\theta \int_{\phi_o}^{\phi_1} d\phi\left\{\left|F(\theta,\phi)+R_1F(\pi-\theta,\phi)\right|^2 + \left|R_2F(\pi-\theta,\phi)\right|^2\right\}$$

The first term in the integrand corresponds to the unde-polarized field, the second term to the depolarized field. The two fields are orthogonal and their energies are additive.

SURFACE WAVES

We observe that the poles of R_1 and R_2 are functions of ψ. Hence we perform the p integration first. While, as mentioned above, our derivation is not meant to be completely rigorous, it is worthwhile to determine carefully whether the poles are actually crossed when the p integration contour is deformed. The whole question of the existence of the surface waves is at stake here. For this purpose, as is customary in similar problems, let us introduce a little conductivity into the medium. That is, we assume that $\mathcal{I}m(k)$ (the imaginary part of k) is slightly positive. The integrands of (22) have branch points at p = k and -k. The integration contour is along the real p axis and we have taken branch cuts as shown in the figure (the same as in ref. [2]).

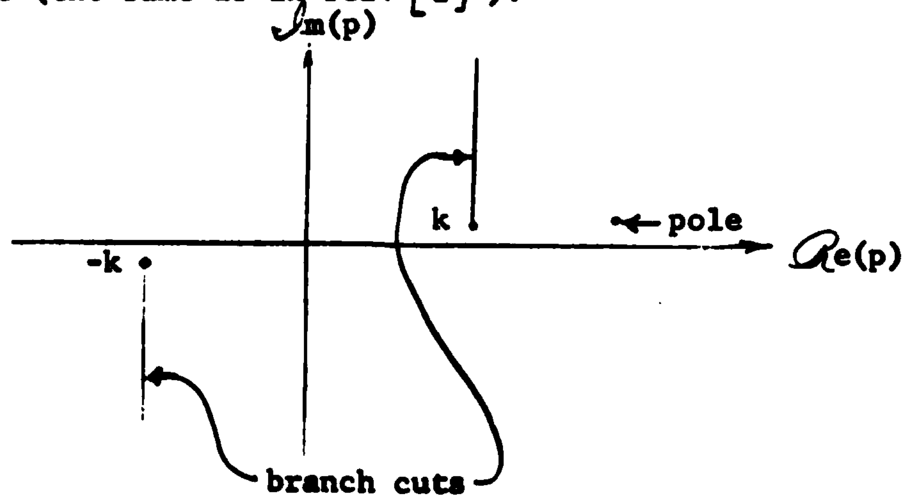


FIGURE NO. 2

We must deform the contour into the upper half plane in order to safely neglect the contribution from infinity. We shall obtain a residue from a pole, therefore, only if it lies in the upper half plane. Let us show that all the poles which are real when $\mathcal{I}m(k) = 0$ do lie in the upper half plane when $\mathcal{I}m(k)$ is positive and approach the real axis as $\mathcal{I}m(k)$ approaches zero. Equations (4), (5), and (13) imply that both X and X_o take on a slight positive imaginary part when k does. Suppose for the moment that ψ is fixed, then (16) defines γ_o as a function of k . When $\mathcal{I}m(k)$ is zero, of course, γ_o is real. Implicit differentiation of (16) with respect to $\mathcal{I}m(k)$ (bearing in mind that all three quantities γ_o, X, and X_o are functions of Im(k)) shows that the derivation of $\mathcal{I}m(\gamma_o)$ with respect to $\mathcal{I}m(k)$ is always positive when γ_o, X, and X_o are real. Thus the introduction of slight conductivity moves the pole (or poles) in the γ-plane from the real axis slightly into the upper half plane. The cut p plane of Figure 2 is mapped into the region lying to the right of the image of the branch cuts in the γ plane (Fig. 3),

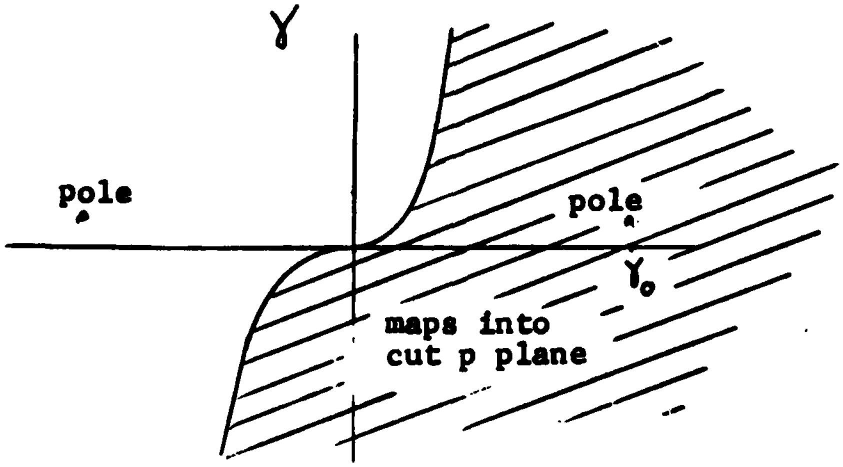


FIGURE NO. 3

Since $p = k\sqrt{1 + \gamma^2}$,

(32) $$\frac{\partial p}{\partial \mathcal{I}m(k)} = \sqrt{1+\gamma^2} + \frac{k\gamma}{\sqrt{1+\gamma^2}} \frac{\partial(\gamma)}{\partial \mathcal{I}m(k)}$$

By setting $\mathcal{I}m(k)$ equal to zero in (32), if γ_o is positive, we see that the left member is positive. Every pole in the plane which maps into the cut p' plane corresponds to positive γ_o and hence maps into the upper half of the p plane. Thus the residues of all poles are actually obtained.

Taking the limit as $\mathcal{I}m(k)$ vanishes, we find from the residue theorem the surface wave terms

(33) $$H_s = i\int_{\phi-\pi/2}^{\phi+\pi/2} \frac{\rho_o}{\gamma_o} U(k\rho_o,\psi)e^{-k\gamma_o z+ik\rho\rho_o\cos(\phi-\psi)}(-r_1\sin\psi$$

$$-i\gamma_o r_2\cos\psi,\ r_1\cos\psi - i\gamma_o r_2\sin\psi,\ r_2\rho_o)d\psi$$

$$E_s = \frac{ik}{\omega\epsilon}\int_{\phi-\pi/2}^{\phi+\pi/2} \frac{\rho_o}{\gamma_o} U(k\rho_o,\psi)e^{-k\gamma_o z+ik\rho\rho_o\cos(\phi-\psi)}(i\gamma_o r_1\cos\psi$$

$$-r_2\sin\psi,\ i\gamma_o r_1\sin\psi + r_2\cos\psi,\ -r_1\rho_o)d\psi,$$

where, we have used our previously defined vector notation and have let ρ_o be the "equivalent index of refraction" of the surface wave:

(34) $$\rho_o = \sqrt{\gamma_o^2 + 1},$$

and

(35) $$r_1 = \text{res}R_1 \Big|_{p=k\rho_o} = \frac{2k\,\gamma_o(X\cos^2\psi + X_o\sin^2\psi + \gamma_o XX_o)}{(2\gamma_o(X\sin^2\psi + X_o\cos^2\psi) - XX_o + 1)\sqrt{1+\gamma_o^2}}$$

$$r_2 = \text{res}R_2 \Big|_{p=k\rho_o} = \frac{2i\sin\psi\cos\psi\,k\,\gamma_o^2(X_o - X)}{(2\gamma_o(X\sin^2\psi + X_o\cos^2\psi) - XX_o + 1)\sqrt{1+\gamma_o^2}}$$

If there is more than one positive value of γ_o satisfying (16) (as there may be in the mixed case $XX_o < 0$), then there is one term of the form (33) for each value of γ_o.

We next evaluate the integrals for large ρ by the method of stationary phase. It is at this point that an important difference enters between the present problem on the one hand, and the two dimensional or isotropic boundary problems on the other hand. The difference is that in the exponential coefficient $k\rho_o\cos(\phi - \psi)$ of ρ, the factor ρ_o depends on ψ. The stationary phase point, thus, in general does not occur at $\psi = \phi$. It occurs at the root of

(36) $$\frac{d}{d\psi}\left(\sqrt{\gamma_o^2 + 1}\;\cos(\phi - \psi)\right) = 0$$

If we call this root $\psi(\phi)$ then, by carrying out the differentiation we find

(37) $$\phi = \psi - \tan^{-1}\left(\frac{\gamma_o\gamma_o'}{1+\gamma_o^2}\right) = \psi - \tan^{-1}\left(\frac{\rho_o'}{\rho_o}\right),$$

where $\gamma_o' = \partial\gamma_o/\partial\psi$, and the principal value of the inverse tangent is implied in (37). Since γ_o is given as a function of ψ by (17), for each ϕ the stationary phase point ψ may be found, in principal, by means of (37). Calculations for simple corrugations ($X_o = 0$) are graphed in Figure 4. It appears possible in certain cases, to have more than one stationary phase point, i.e., two or more values of ψ for a given ϕ.* We shall consider first those cases where only a single stationary phase point occurs.

For the stationary phase evaluation we also require

(38) $$s(\phi) = \frac{\partial^2}{\partial\psi^2}\left(\sqrt{1+\gamma_o^2}\,\cos(\phi-\psi)\right)\Bigg|_{\psi=\psi(\phi)}$$

$$= \left\{\frac{\gamma_o'^2 - (1+\gamma_o^2)^2 + \gamma_o\,\gamma_o''(1+\gamma_o^2) - 2\,\gamma_o^2\,\gamma_o'^2}{\left[(1+\gamma_o^2)((1+\gamma_o^2)^2 + \gamma_o^2\,\gamma_o'^2)\right]^{1/2}}\right\}$$

We obtain finally the surface wave field as

(39) $$H_s = \frac{(1+i)\rho_o}{\gamma_o}\sqrt{\frac{2\pi}{k\rho s}}\;U(k\rho_o,\psi)\,e^{-\gamma_o kz+ik\rho\rho_o\cos(\phi-\psi)}\,(-r_1\sin\psi - i\,\gamma_o r_2\cos\psi,\;\; r_1\cos\psi - i\,\gamma_o r_2\sin\psi,\;\; r_2\rho_o)$$

$$E_s = \frac{(1+i)\rho_o k}{\omega\epsilon\,\gamma_o}\sqrt{\frac{2\pi}{k\rho s}}\;U(k\rho_o,\psi)\,e^{-\gamma_o kz+ik\rho\rho_o\cos(\phi-\psi)}\,(i\gamma_o r_1\cos\psi - r_2\sin\psi,\;\; i\,\gamma_o r_1\sin\psi + r_2\cos\psi,\;\; -r_1\rho_o)$$

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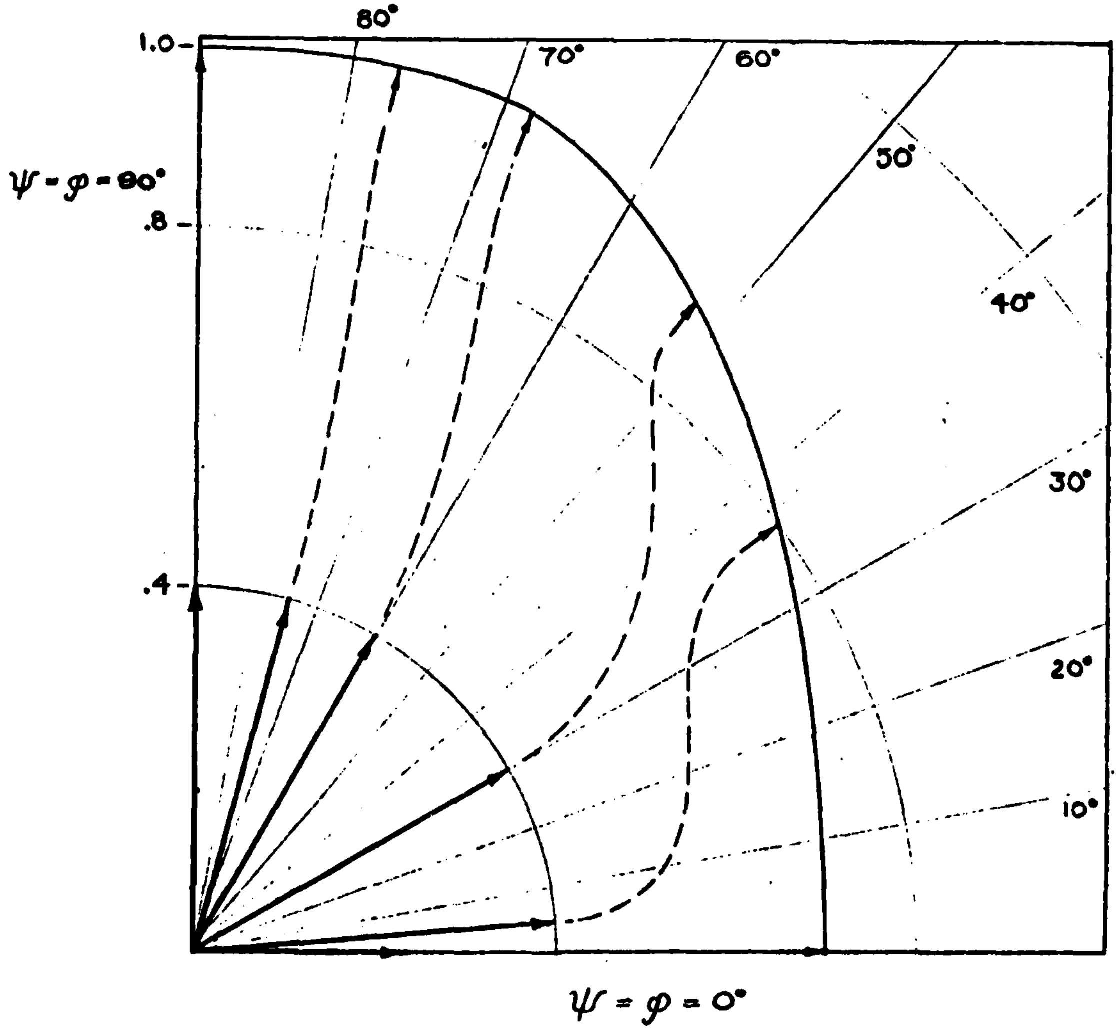
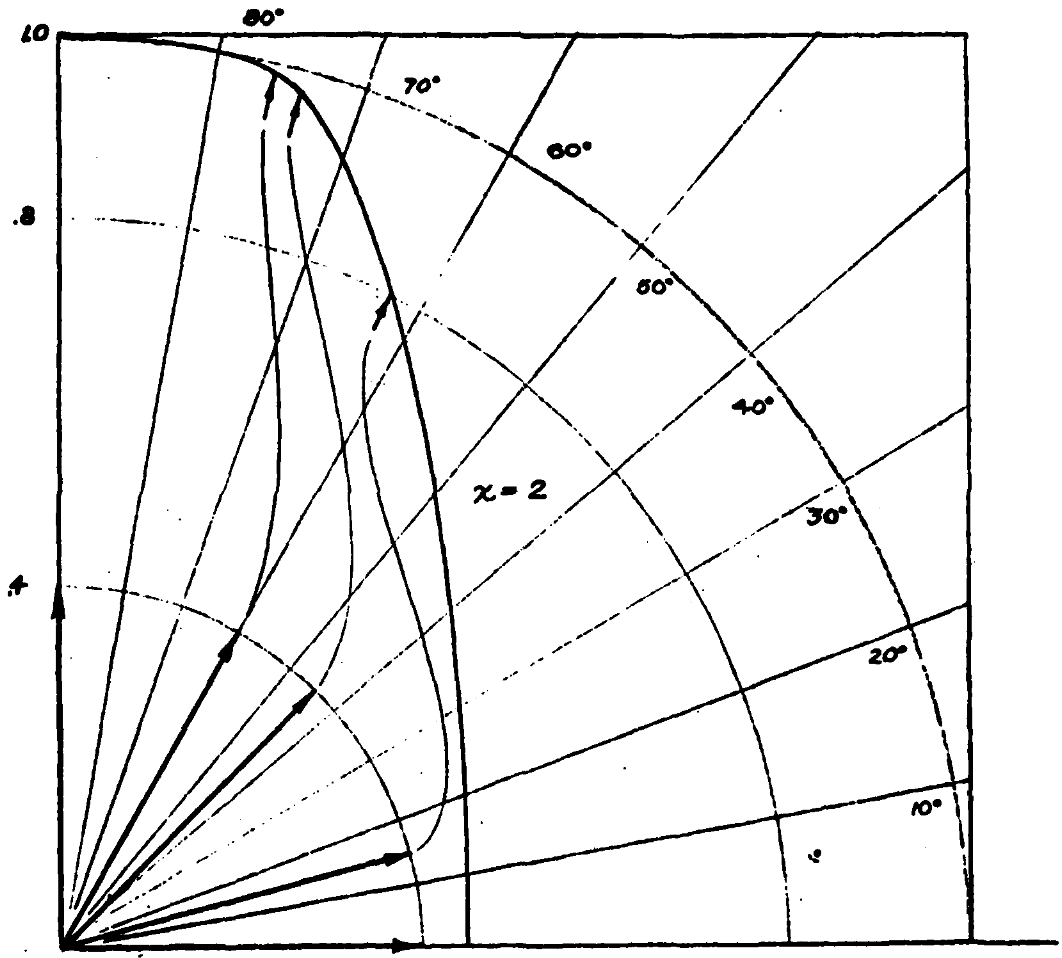
* This is shown in Figure 7 for $X = 1$, $X_o = -4$.



$$r(\varphi) = \frac{h}{\sqrt{1+\zeta^2}\ \cos(\varphi-\psi)}$$

FIGURE 4a

INDUCTIVE CASE: X = 1, X_o = 0. Showing Far Field Wave Front and Relation of Initial Energy Flow Direction ψ and Far Field Energy Flow $\emptyset$. (Isotropic Phase Source)

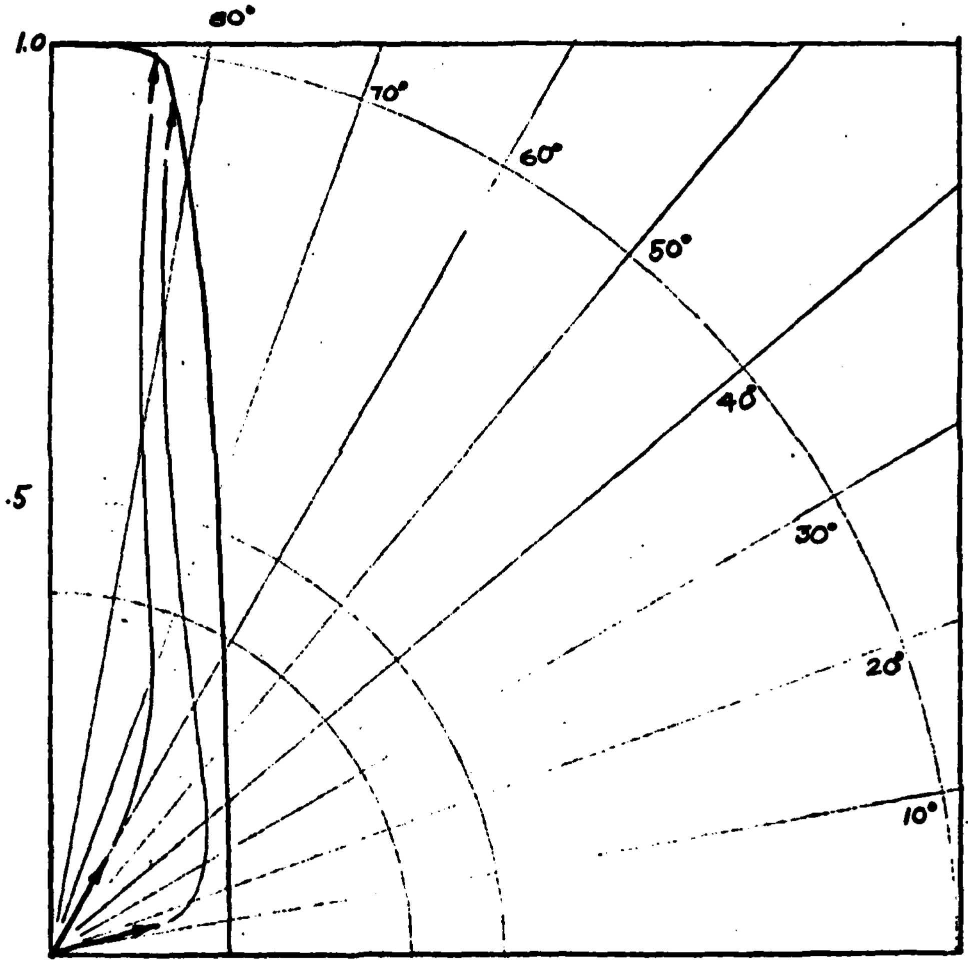


$\chi = 2$

$$r(\varphi) = \frac{k}{\sqrt{1+\gamma_0^2}\,\cos(\varphi-\psi)}$$

FIGURE 4b

INDUCTIVE CASE: X = 2, X_o = 0. Showing Far Field Wave Front and Relation of Initial Energy Flow Direction ψ and Far Field Energy Flow ϕ. (Isotropic Phase Source)



$\chi = 5$

$$r(\varphi) = \frac{k}{\sqrt{1 + \gamma_0 \cos(\varphi - \psi)}}$$

FIGURE 4c

INDUCTIVE CASE: X = 5, X_0 = 0. Showing Far Field Wave Front and Relation of Initial Energy Flow Direction ψ and Far Field Energy Flow ϕ. (Isotropic Phase Source)

The power flow between the meridians $\phi_o \angle \phi \angle \phi_1$ is

$$(40) \quad P_s = \frac{\mathcal{R}e}{2}\left\{\int_o^{\infty} dz \cdot \int_{\phi_o}^{\phi_1} \rho d\phi (E_s \times \bar{H}_s)\right\} =$$

$$\frac{\pi}{2k\omega\epsilon}\int_{\phi_o}^{\phi_1} \frac{\rho_o^3}{|s|\,\gamma_o^3}\,|U(k\rho,\psi)|^2 d\phi \Big((r_1{}^2-r_2^2)\cos\psi - 2i\,\gamma_o r_1 r_2 \sin\psi,$$

$$(r_1{}^2-r_2^2)\sin\psi + 2i\,\gamma_o r_1 r_2 \cos\psi,\ 0\Big)$$

where ψ is defined as a function of ϕ by (37). The surface wave power flow is thus parallel to the surface. The radial component over an element $d\phi$ is

$$(41)\ P_{s\rho} = \frac{\pi\rho_o^3|U(k\rho,\psi)|^2}{2k\omega\epsilon|s|\gamma_o^3}\Big((r_1^2-r_2^2)\cos(\phi-\psi)+2i\gamma_o r_1 r_2 \sin(\phi-\psi)\Big)$$

The azimuthal or ϕ-component is

$$(42)\ P_{s\phi} = \frac{\pi\rho_o{}^3|U(k\rho,\psi)|^2}{2k\omega\epsilon|s|\,\gamma_o^3}\Big((r_2{}^2-r_1{}^2)\sin(\phi-\psi)+2i\gamma_o r_1 r_2 \cos(\phi-\psi)\Big).$$

By direct but lengthy computation (details in Appendix I), we can prove that

$$(43) \quad (r_2{}^2-r_1{}^2)\sin(\phi-\psi) + 2i\gamma_o r_1 r_2 \cos(\phi-\psi) \equiv 0.$$

As a consequence,

(44) $$P_{s\phi} = 0$$

and

(45) $$P_{s\rho} = P_s = \frac{\pi \rho_o^3 \left|U(k\rho,\psi)\right|^2 (r_1^2 - r_2^2)}{2k\omega\epsilon \left|s\right| \gamma_o^3 \cos(\phi-\psi)}$$

Equation (44) implies that the power flow is always radial. The direction of power flow is, by definition, the direction of the ray. Thus the rays corresponding to the surface waves are straight lines. One can also show that the extremals for least time in an anisotropic, homogeneous medium are straight lines, so that the rays not only denote the direction of power flow but also are the quickest possible paths for energy flow. Fermat's principal is thereby verified for this case.

However, the rays are not normal to the wave fronts. The wave fronts are defined as in the isotropic case either as (a) the contours of constant phase or (b) solutions to the eikonal equation. We proceed now to find the equation of the wave fronts and to show that these two definitions are equivalent in the present case. For simplicity we suppose that the phase of $U(k\rho_o, \psi)$ is independent of ψ, as in the case of a magnetic dipole source.

At a fixed instant of time (or fixed value of ρ) the surface wave traveling in direction ψ has advanced from the origin a distance ρ / ρ_o. Figure 5 shows, for example, four neighboring phase fronts. The set of all these phase fronts form the envelope of a curve C which is the wave front. To prove that C is a wave front, first repeat the process just described for a slightly larger value of ρ, say $\rho + \Delta\rho$. There results a second enveloped curve C_1. To show that both C and C_1 are wave fronts, observe that the distance between them measured at any point along the normal to either of them is (correct to first order in $\Delta\rho$) $\Delta\rho/\beta_0$, where β_0 is evaluated for the direction ψ of this normal. This establishes the wave fronts as a set of curves whose orthogonal trajectories or wave normals separate any two nearby curves by distances proportional to the wave length in the normal directions. This means that they satisfy the eikonal equation. We proceed to derive the equation of the wave fronts:

The equation of the phase front traveling in direction ψ is

(46) $$x\cos\psi + y\sin\psi = \rho/\beta_0 .$$

These lines are the envelope of the wave front.

The wave front is thus given by the intersection of (46) and its derivative:

(47) $$-x\sin\psi + y\cos\psi = \frac{-\rho}{\beta_0^2}\beta_0' .$$

The equation of the wave front is therefore

(48) $$x = \frac{\rho}{\beta_0}\cos\psi + \frac{\rho\beta_0'}{\beta_0^2}\sin\psi$$

$$y = \frac{-\rho}{\beta_0^2}\beta_0'\cos\psi + \frac{\rho}{\beta_0}\sin\psi .$$

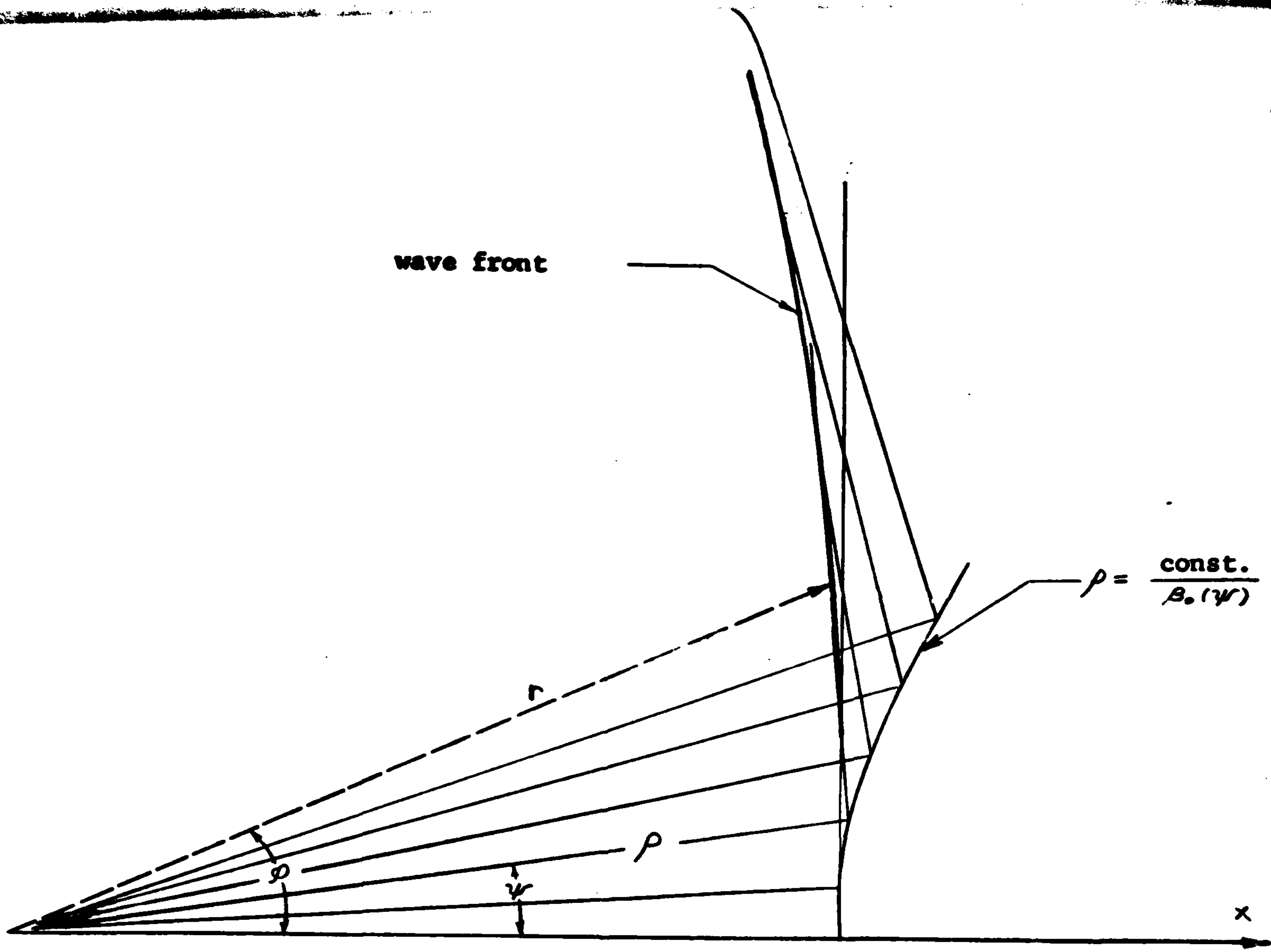


FIGURE 5

Showing Wave Front, given in polar coordinates by $r\beta_o\cos(\phi-\psi)$ = const., as Envelope of Plane Phase Fronts. Each Phase Front Propagates away from the Origin with its Characteristic Phase Velocity

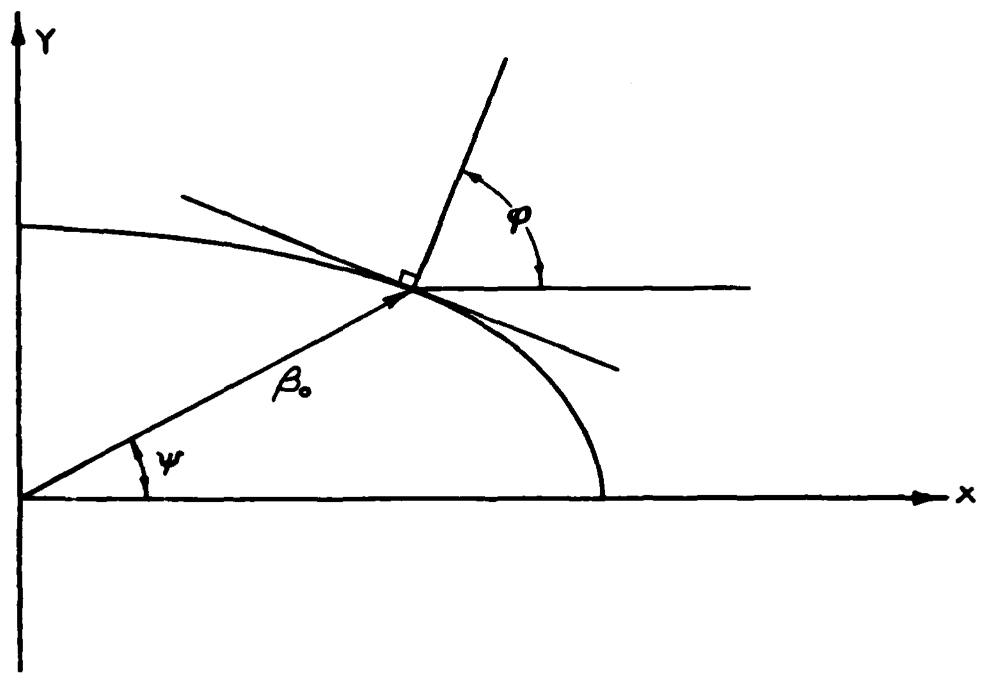


FIGURE 6

Showing Another Relationship between ϕ and ψ. Curve is β_o vs ψ in Polar Coordinates. Normal to Curve makes Angle ϕ with Positive x-axis, since

$$\phi = \psi - \tan^{-1} \frac{\beta_o'}{\beta_o}$$

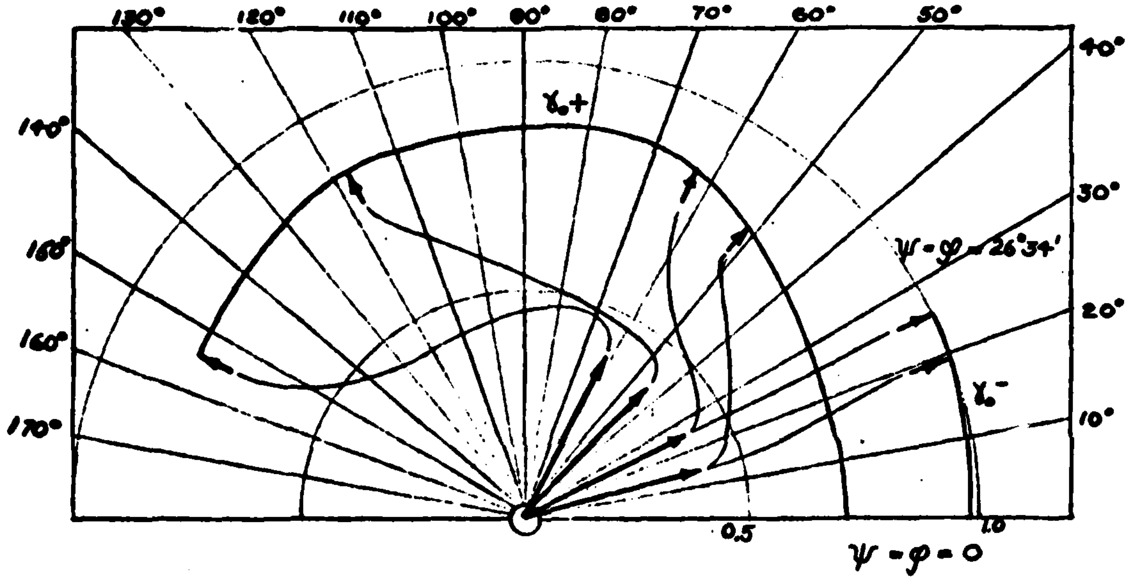


FIGURE 47

MIXED CASE: $X = 1$, $X_o = -4$. Showing Two Wave Fronts with Different Phase Velocities γ_o+ and γ_o-. Isotropic Source Radiating into First Quadrant Only.

If we let (r,ϕ) be the polar coordinates of the wave front then setting $x = r\cos\phi$, $y = r\sin\phi$ in (48) we find

(49) $$r = \frac{\rho}{\beta_o^2}\sqrt{\beta_o^2 + \beta_o'^2}$$

and

(50) $$\tan\phi = \frac{\beta_o \tan\psi + \beta_o'}{\beta_o + \beta_o' \tan\psi} .$$

The latter may be rewritten as

(51) $$\tan(\phi-\psi) = \frac{-\beta_o'}{\beta_o} .$$

Comparison of (51) with (37) shows that ϕ is the same as the stationary phase observation angle. Also (51) implies that

(52) $$\cos(\phi-\psi) = \frac{\beta_o}{\sqrt{\beta_o^2 + \beta_o'^2}}$$

so that (49) may be written as

(53) $$r\beta_o\cos(\phi-\psi) = \rho .$$

Comparison with (39) shows that the wave front given by (48) in rectangular coordinates or (49) in polar coordinates is also the contour of constant phase in the far field of the surface wave. Definitions (a) and (b) are therefore both verified.

COMPARISON WITH PROPAGATION IN A CRYSTALLINE MEDIUM

It is instructive to compare the propagation of surface waves over an anisotropic boundary with the propagation of plane waves in a homogeneous, anisotropic medium such as a crystalline medium. The latter is described in considerable detail in Planck's "Theory of Light" [3] .

One basic difference between the two phenomenon is that the propagation directions of the surface waves all lie in the plane of the surface, while many of the interesting features of the crystalline propagation (such as the existence of the primary and secondary optical axes) depend on the three dimensional properties of the anisotropic medium. If we restrict the latter to a two dimensional medium, however, some interesting comparisons can be made. In each case, a single infinite plane wave propagates energy in a direction which does not coincide with the normal to the wave front unless the latter is in a principal direction. In each case, energy from a point source propagates along the radial lines from the source although the wave fronts are not concentric circles. However, the dependence of phase velocity on direction is more complicated in the surface wave case than in the crystal case. While in the crystalline medium the wave fronts are ellipses; on the anisotropic surface the wave fronts are more complicated curves. Examples of the latter are shown in Figures 4a, b, c and 7. In the crystalline medium, apart from certain exceptional directions, exactly two waves can and do propagate from a source. These are cross polarized from each other, and have different phase velocities. In each wave E and H are perpendicular to S the direction of energy flow. On the other hand, on the anisotropic surface, in each direction either zero, one, or two waves can propagate each with its own phase velocity. Each wave is a hybrid mode, a combination of TE and TM modes. Each of the latter is a separate solution to Maxwell's equations but only a certain proper combination of the two satisfies the boundary conditions. This same combination guarantees that the surface wave energy flow from a point source will be radial (i.e., that Equation (43) holds). The energy in the hybrid mode is not the sum of the separate energies in the TE and TM parts so that there is energy coupling between the latter. E and H are not usually perpendicular to S.

The radiated field reflected from the anisotropic boundary, on the other hand, has none of the properties of propagation in an anisotropic medium. For the radiated field, E , H, and S are an orthogonal set and S is perpendicular to the wave front which is spherical. While the radiated field generally has both TE and TM modes (even if the source is pure TE or pure TM), the power flows of these modes are orthogonal to each other.

EFFICIENCY IN A PRINCIPAL DIRECTION COMPARED WITH CULLEN

When $\psi = 0$, then from (17), $\gamma_0' = 0$ and, from (37), $\phi = 0$. We observe from (16) that $\gamma_0 = X$ or $-1/X_0$ (if these are positive). Substitution of $-1/X_0$ for γ_0 and zero for ψ

into (35) yields $r_1 = r_2 = 0$. Hence there is no excitation of the mode for which $\gamma_o = -1/X_o$ in the direction $\psi = 0$ even if this mode appears to be a free solution of Maxwell's equations and the boundary condition. For the $\gamma_o = X$ mode at $\psi = 0$,

(54) $$r_1 = \frac{2kX^2}{\sqrt{1+X^2}} , \quad r_2 = 0.$$

From (17), correct to second order in ψ

(55) $$\gamma_o = X + \frac{\psi^2 (X_o - X)(1+X^2)}{1+XX_o}$$

So that from (37), correct to first order in ψ

(56) $$\phi = \psi \left(1 + \frac{2X(X-X_o)}{1+XX_o}\right).$$

From (38)

(57) $$s(\phi) = -\sqrt{1+X^2}\left(1 + \frac{2X(X-X_o)}{1+XX_o}\right)$$

Finally from (45)

(58) $$P_s = \frac{2\pi kX \left| U(k/\sigma_o, 0)\right|^2}{\omega\epsilon\left(1 + \frac{2X(X-X_o)}{1+XX_o}\right)} .$$

For the purpose of comparing our result with Cullen's [1] let us define the efficiency η of surface wave excitation in a given direction ϕ as the ratio of surface wave power radiated outward between the meridian ϕ and $\phi + d\phi$ to the total power radiated outward between these meridians in the limit as $d\phi$ approaches zero. From (31) and (57), we see that for $\phi = 0$

$$(59)\qquad \eta = \frac{P_s/P_r}{1 + P_s/P_r}$$

where

(60)

$$\frac{P_s}{P_r} = \frac{4\pi X\left|U(k\beta_o,0)\right|^2}{\left(1+\frac{2X(X-X_o)}{1+XX_o}\right)\int_o^{\pi/2}\sin\theta d\theta\left|F(\theta,0)+R_1\,F(\pi-\theta,0)\right|^2},$$

where

$$(61)\qquad R_1 = 1 + \frac{2X(iX_o\cos\theta-1)}{X_o\cos^2\theta - i\cos\theta(XX_o-1)+X}.$$

Apart from the fact that Cullen's formula for efficiency applies only to the case $X_o = 0$ (single corrugations), there are three differences between our formula and his.

(1) A factor c- $1+X^2$ is missing from the denominator of (60). The absence of this factor appears to be an inherent difference between the two and three dimensional problems, since its counterpart is also missing in the rigorous expression (130) [2] for the surface wave in the three dimensional isotropic case. The difference between the two and three dimensional problems in this respect may be traced back to the difference between the three dimensional integral representation for the field as given by (22) and Cullen's integral representation (19) of [1]. The expressions are almost the same, but the factor p in the integrand of (22) has no counterpart in (19) of [1]. This factor p is the Jacobean of the change of coordinates in the transform plane from

rectangular, with area element dudv, to polar, with area element $\rho d\rho d\psi$. (This change of coordinates is carried out explicitly in [2]). The same change of coordinates in the two dimensional case need never be explicitly considered since it degenerates to the identity transformation.

When the residue theorem is applied, p is replaced by ρ_0 which, at $\phi = 0$, equals $\sqrt{1+X^2}$. When the latter factor is squared for the computation of power, the factor of $1+X^2$, which appears in the numerator of Cullen's expression, (and would otherwise also appear in (60)) is cancelled (Note that Cullen uses ρ for our X).

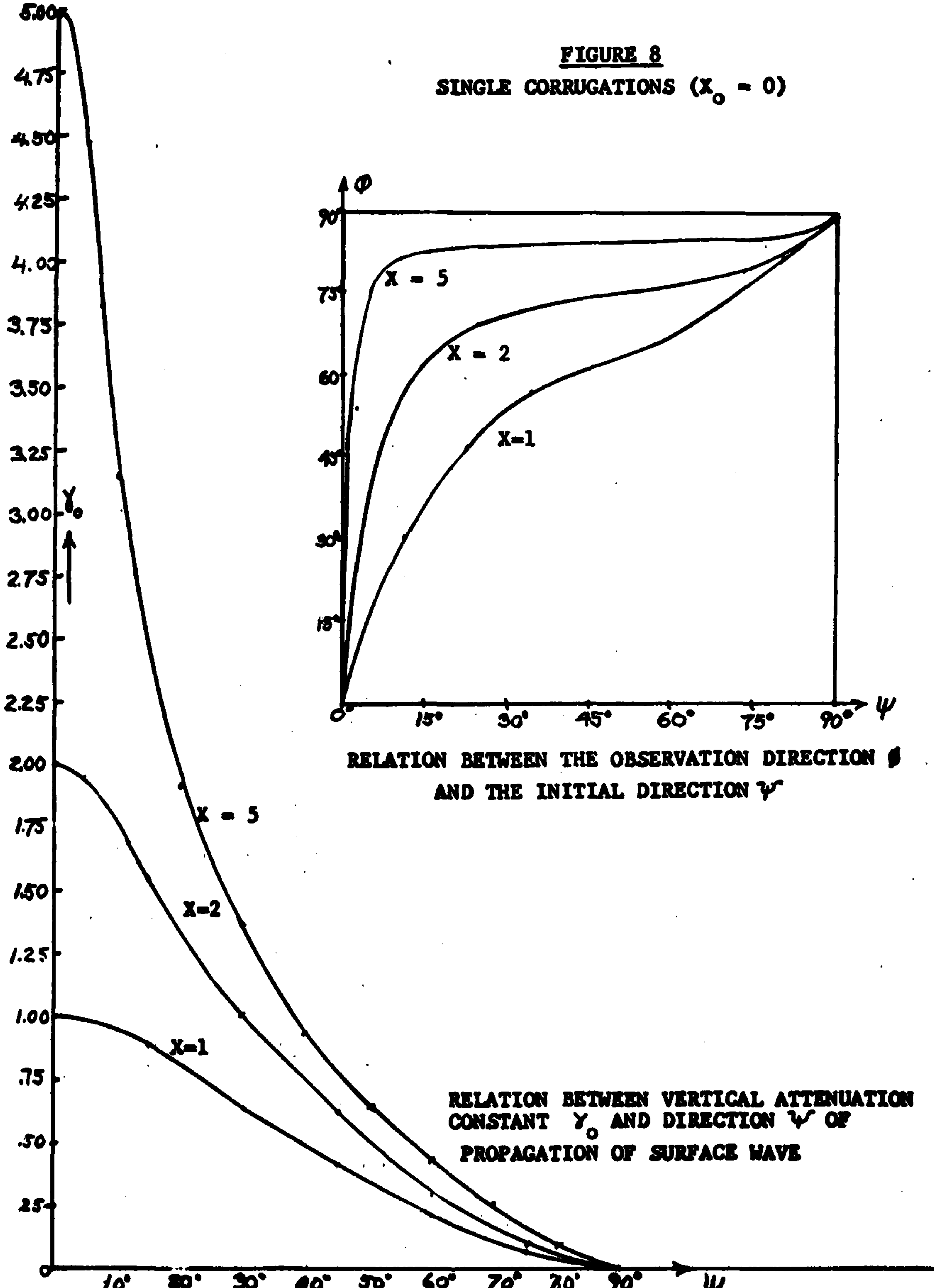
(2) In computing total radiated power, we must integrate over a large sphere, not over a cylinder as in Cullen's case, so that the factor $\sin\theta$ appears in the integrand in (60) but does not appear in the analogous expression (35) of [1]. Again this factor is a Jacobean which is not the same in two and three dimensions.

(3) The factor $1 + \frac{2X(X-X_0)}{1+XX_0}$ appearing in the denominator of (60) does not appear in Cullen's expression. It arises from the fact that the wave number $k\rho_0$ is a function of direction ψ, i.e., that the boundary is anisotropic, so that s is a function of ϕ. In Cullen's case s equals $-\rho_0$ and is independent of ϕ. To say the same thing in more physical though perhaps imprecise terms: the fact that the wave fronts of a point source are not circles implies that,

in the stationary phase evaluation, the far field will vary with the curvature of the wave front at the stationary phase point - the flatter the phase front and the more "stationary" the stationary phase point, the larger the contribution to the far field. This phenomenon has no analogue in the isotropic boundary case.

FIGURE 8

SINGLE CORRUGATIONS (X_o = 0)



RELATION BETWEEN THE OBSERVATION DIRECTION φ AND THE INITIAL DIRECTION ψ

RELATION BETWEEN VERTICAL ATTENUATION CONSTANT γ_o AND DIRECTION ψ OF PROPAGATION OF SURFACE WAVE

APPENDIX I

Proof of (43):

It is sufficient to prove

(A1) $$\tan(\phi-\psi) = \frac{2i\,\gamma_o r_1 r_2}{r_1^2 - r_2^2},$$

or in view of (37)

(A2) $$\frac{-\gamma_o'}{1+\gamma_o^2} = \frac{2ir_1r_2}{r_1^2-r_2^2}.$$

From (35)

(A3) $$\frac{2ir_1r_2}{r_1^2-r_2^2} = \frac{2SC(X-X_o)\,\gamma_o(XC^2-X_oS^2+\gamma_oXX_o)}{(XC^2+X_oS^2+\gamma_oXX_o)^2 + S^2C^2\gamma_o^2(X-X_o)^2},$$

where we abbreviate

(A4) $$S = \sin\psi, \qquad C = \cos\psi.$$

Since γ_o is the root of the denominator in (11) we differentiate this denominator to show that

(A5) $$\gamma_o'(2\gamma_o(X_oC^2+XS^2) + 1-XX_o)+2(1+\gamma^2)(X-X_o)SC = 0.$$

This implies that

(A6) $$\frac{-\gamma_o'}{1+\gamma_o^2} = \frac{2(X-X_o)SC}{2\gamma_o(X_oC^2+XS^2)+1-XX_o} = \frac{2(X-X_o)SC}{\pm\sqrt{(1+XX_o)^2+(X-X_o)^2\sin^2 2\psi}},$$

where the sign alternative is to be chosen to insure equality. We find then that (A2) will hold if and only if

(A7) $$\gamma_o(XC^2+X_oS^2+\gamma_oXX_o)(2\gamma_o(X_oC^2+XS^2)+1-XX_o) =$$

$$(XC^2+X_oS^2+\gamma_oXX_o)^2 + S^2C^2\gamma_o^2(X-X_o)^2,$$

(A7) may be written as

(A8)

$$\gamma_o^3(2XX_o)(X_oC^2+XS^2)+\gamma_o^2\left(XX_o-2(XX_o)^2+2(X_oC^2+XS^2)(XC^2+X_oS^2)\right.$$

$$\left.-S^2C^2(X-X)^2\right)+\gamma_o(XC^2+X_oS^2)(1-3XX_o)\ -\ (XC^2+X_oS^2)^2=0\ .$$

Since from (16)

(A9) $\gamma_o^2(X_oC^2+XS^2) = XC^2+X_oS^2 + \gamma_o(XX_o-1)$,

we may substitute for the cubic term in γ_o in (A8) reducing this to terms of second degree or less, which upon combination with the remaining terms of (A8) yield

(A10) $\gamma_o^2((X_o^2+X^2)S^2C^2+XX_o(2C^4+2S^4+2S^2C^2-1)) + (1-XX_o)\gamma_o(XC^2+X_oS^2)$

$$-(XC^2+X_oS^2)^2=0$$

The coefficient of γ_o^2 may be factored by $(XC^2+X_oS^2)$, and with the cancellation of the latter, (A10) reduces to (A9). Since all the algebra is reversible, we have proved (43).

PART II

EXTENSION OF SOME RESULTS OF CULLEN ON "THE EXCITATION OF PLANE SURFACE WAVES"

In [1], Cullen has obtained a formula for the excitation efficiency of surface waves launched over a metal clad dielectric. The formula is based on the following assumptions.

(a) the dielectric thickness d is small compared to the wavelength λ,

(b) the energy carried by the surface wave in the dielectric is negligible compared to the energy carried by the surface wave in the space above,

(c) an infinite horizontal magnetic line current is the source.

For one application encountered at Technical Research Group it was of interest to obtain a formula generalizing Cullen's which drops the first restriction and replaces his source by a finite horizontal magnetic line current. A restriction on thickness is still invoked namely

(1) $$d < \lambda/\left(2\sqrt{\epsilon-1}\right)$$

but this is used to insure that only a single surface wave can propagate. When the thickness is not small compared to the wavelength, the assumption that the energy carried by the surface wave in the dielectric is negligible is no longer tenable and it is dropped here. It is rather surprising to discover that despite the apparent immediate increase of complexity caused by the waiver of these assumptions, our final formula for excitation efficiency is hardly more complicated than Cullen's.

<u>This section is not in the least self contained</u>. Rather we simply point out, line by line, what changes must be made in Cullen's derivation when assumptions (a) and (b) are not made, and, as a first step, we obtain an intermediate formula when only these assumptions are dropped. In order to change assumption (c), we then rely on the results of the three dimensional analysis given in reference [2] and in the first part of this paper.

We use on the whole Cullen's notation. The first exception is that we let ϵ be the relative dielectric constant of the dielectric. However, in contrast to the first part of this paper, now the implicit time dependence is e^{jwt} and θ is the grazing angle rather than the incidence angle.

<u>REMOVAL OF ASSUMPTIONS (a) and (b)</u>

The reflection coefficient for a TM plane wave grazing the metal clad dielectric surface at angle θ is

$$R = \frac{j\sin\theta + \delta(\cos\theta)}{j\sin\theta - \delta(\cos\theta)}$$

where

$$\delta(t) = \frac{\sqrt{\epsilon - t^2}}{\epsilon}\tan\left(kd\sqrt{\epsilon - t^2}\right) \tag{2}$$

Section (10) of Cullen may now be read through expression (44) if the latter is replaced by $k\delta(\cos\theta)$. Cullen's formula for the radiated power (34) remains correct with his δ replaced by $\delta(\cos\theta)$.

In order to find the formula for the surface wave power, observe that equation (19) of [1] remains correct if k is replaced by $k\zeta(\beta/k)$. The generalized reflection coefficient

(3) $$R(\beta) = \frac{u+k\zeta(\beta/k)}{u-k\zeta(\beta/k)}$$

then appears in (19). Its residue at the pole $u = k_1$, where $\beta = \beta_1 = \sqrt{k^2+k_1^2}$, is

(4) $$\frac{2\epsilon k_1^2(k^2\epsilon-\beta_1^2)}{\beta_1(\epsilon-1)\left[\epsilon k^2+k_1 d(k_1^2(\epsilon+1)+k^2)\right]} .$$

The leading term of (20) [1] is then

(5) $$\frac{\dfrac{-2k}{\beta_1}\sqrt{\dfrac{\chi_o}{\mu_o}}\; Ve^{-k_1(y+h)-j\beta_1 x}\;\epsilon k_1(k^2\epsilon-\beta_1^2)}{(\epsilon-1)\left[\epsilon k^2 + k_1 d(k_1^2(\epsilon+1)+k^2)\right]}$$

The power radiated in the surface wave, is then (instead of (36)[1])

(6) $$P_s = \frac{4k\sqrt{\dfrac{\chi_o}{\mu_o}}\; V^2 e^{-2k_1 h}\;\epsilon^2 k_1^2(k^2\epsilon-\beta_1^2)^2}{\beta_1(\epsilon-1)^2\left[k^2\epsilon+k_1 d(k_1^2(\epsilon+1)+k^2)\right]^2}\left\{I_1+I_2\right\},$$

where

(7) $$I_1 = \int_0^\infty e^{-2k_1 y}\,dy = 1/2k_1 ,$$

(8) $$I_2 = \frac{1}{\epsilon}\int_{-d}^{0}\left[\frac{\cos y\sqrt{k^2\epsilon-\beta_1^2}}{\cos d\sqrt{k^2\epsilon-\beta_1^2}}\right]^2 dy =$$

$$\frac{d\left(1+\dfrac{\sin(2d\sqrt{k^2\epsilon-\beta_1^2})}{2d\sqrt{k^2\epsilon-\beta_1^2}}\right)}{2\epsilon\cos^2(d\sqrt{k^2\epsilon-\beta_1^2})} .$$

Let us now introduce a simplifying notation

(9) $\beta_1 = kn, \quad k_1 = k\sqrt{n^2-1}$,

where n is the equivalent "index of refraction". The surface wave existence condition (43) of [1] may be written as

(10) $\epsilon\sqrt{n^2-1} = \sqrt{\epsilon-n^2}\tan(kd\sqrt{\epsilon-n^2})$,

so that (8) becomes

(11) $I_2 = \dfrac{d(\epsilon-1)(n^2(\epsilon+1)-\epsilon) + (\epsilon/k)\sqrt{n^2-1}}{2\epsilon(\epsilon-n^2)}$.

The ratio of the power in the surface wave carried above the dielectric to the power carried in the dielectric is

(12) $\dfrac{I_1}{I_2} = \dfrac{\epsilon(\epsilon-n^2)}{\sqrt{n^2-1}\left(kd(\epsilon-1)\left(n^2(\epsilon+1)-\epsilon\right) + \epsilon\sqrt{n^2-1}\right)}$.

When $kd \ll 1$ so that $n^2-1 \ll 1$ this ratio is large, and the term I_2 may be neglected in (6) compared to I_1. In the general case, however, I_2 is not negligible, and (6) must be written as

(13) $$P_s = \frac{2k\sqrt{\frac{K_o}{\mu_o}}\, V^2 e^{-2kh\sqrt{n^2-1}}\, \epsilon\sqrt{n^2-1}(\epsilon-n^2)}{n(\epsilon-1)\left[\epsilon+(kd\sqrt{n^2-1})\;(n^2(\epsilon+1)-\epsilon)\right]}$$

From (35) of [1] the radiated power is

(14) $$P_R = \frac{2k}{\pi}\sqrt{\frac{K_o}{\mu_o}}\; V^2 \int_0^{\pi/2} F^2(\theta)d\theta$$

and the efficiency is

(15) $$\eta = \frac{P_s}{P_R + P_S}$$

where

$$\text{(16)} \qquad \frac{P_S}{P_R} = \frac{\pi e^{-2kh\sqrt{n^2-1}} \; \epsilon\sqrt{n^2-1}(\epsilon-n^2)}{n(\epsilon-1)\left[\epsilon+(kd\sqrt{n^2-1})(n^2(\epsilon+1)-\epsilon)\right]\int_0^{\pi/2} F^2(\theta)d\theta} .$$

FINITE LINE SOURCE

Two changes must be made in formula (16) if the line source is finite instead of infinite. Consider any source whose three dimensional far field pattern in free space is uniform in the xy plane, i.e., independent of angle, and has H perpendicular to this plane. The analysis of [2] - or of the preceeding section - shows that an extra factor of n appears in the surface wave field and hence n^2 in the surface wave power. In the radiation field the power flow must be integrated over a large sphere whose area element is ordinarily $\sin\theta d\theta d\phi$ - but since θ is now the grazing angle instead of the incidence angle in the present notation - this factor is $\cos\theta d\theta d\phi$ here. The $d\phi$ cancels the same factor in the surface wave power expression, if we consider the ratio of power flows per unit azimuthal angle. The correct expression for efficiency is thus (15) with (16) replaced by

$$\text{(17)} \qquad \frac{P_S}{P_R} = \frac{\pi n\epsilon\sqrt{n^2-1}\;(\epsilon-n^2)e^{-2kh\sqrt{n^2-1}}}{(\epsilon-1)\left[\epsilon+(kd\sqrt{n^2-1})(n^2(\epsilon+1)-\epsilon)\right]\int_0^{\pi/2}\cos\theta F^2(\theta)d\theta} .$$

Both additional factors in the three dimensional case tend to enhance the power in the surface wave at the expense of power in the radiation field when compared to the two dimensional case.

It is also well to point out that the preceeding formulas for efficiency will in general be gravely in error if the source is not symmetric with respect to a plane parallel to the ground plane, and can be considerably in error if the source is anything but isotropic in the xy plane. This is especially true for those high values of efficiency which are primarily due to the cancelation of the directed and reflected rays in the radiation field. (See Fig. 10 of [1]).

Figure 9 shows the two and three dimensional efficiencies for four values of ϵ vs h/λ and d/λ, the height of the source and the thickness of the dielectric, respectively, in wavelengths. The corresponding values of n^2 are shown. The values of h/λ where the radiated field is zero at both 45° and 60° grazing angle are shown. These are not necessarily the most efficient heights. If the conclusions of reference [5] were to be taken literally for the dielectric clad ground plane then at the former height the efficiency would always be 100%. Cullen's conclusions based on the case $\epsilon = 2.56$, $n^2 = 1.25$ also appear not to be highly relevant for other values. The optimum height for maximum efficiency is apparently often zero. Moreover, the optimum height, which is always a small fraction of a wavelength, is not critically different from zero. The efficiency does, however, depend critically on d/λ (or n). None of these conclusions are apparent from Cullen's work.

Efficiency η of Excitation of Surface Wave over Dielectric Slab of Thickness d/λ for Magnetic Dipole at Height h/λ above Dielectric

$\epsilon = 2.56$

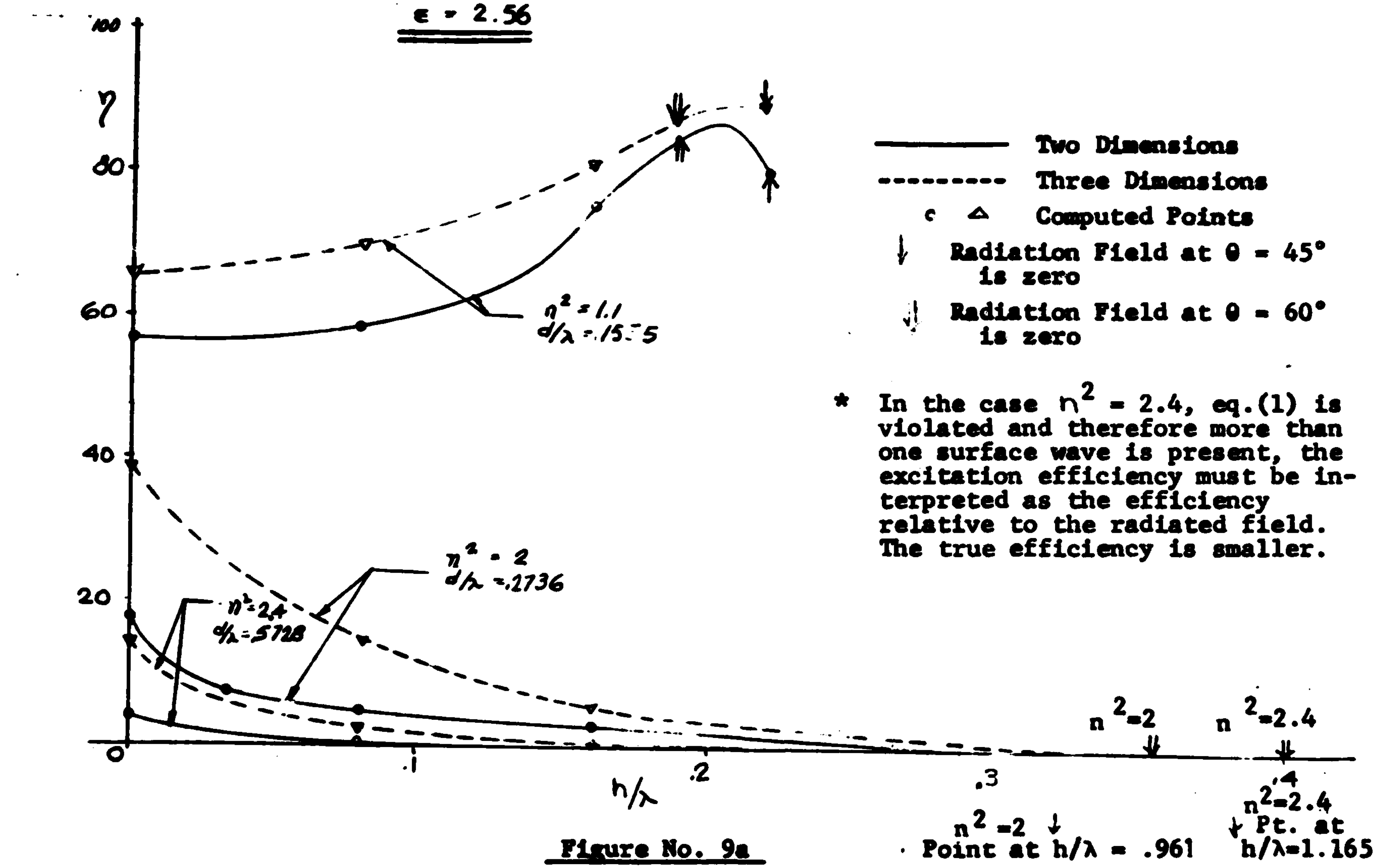


Figure No. 9a

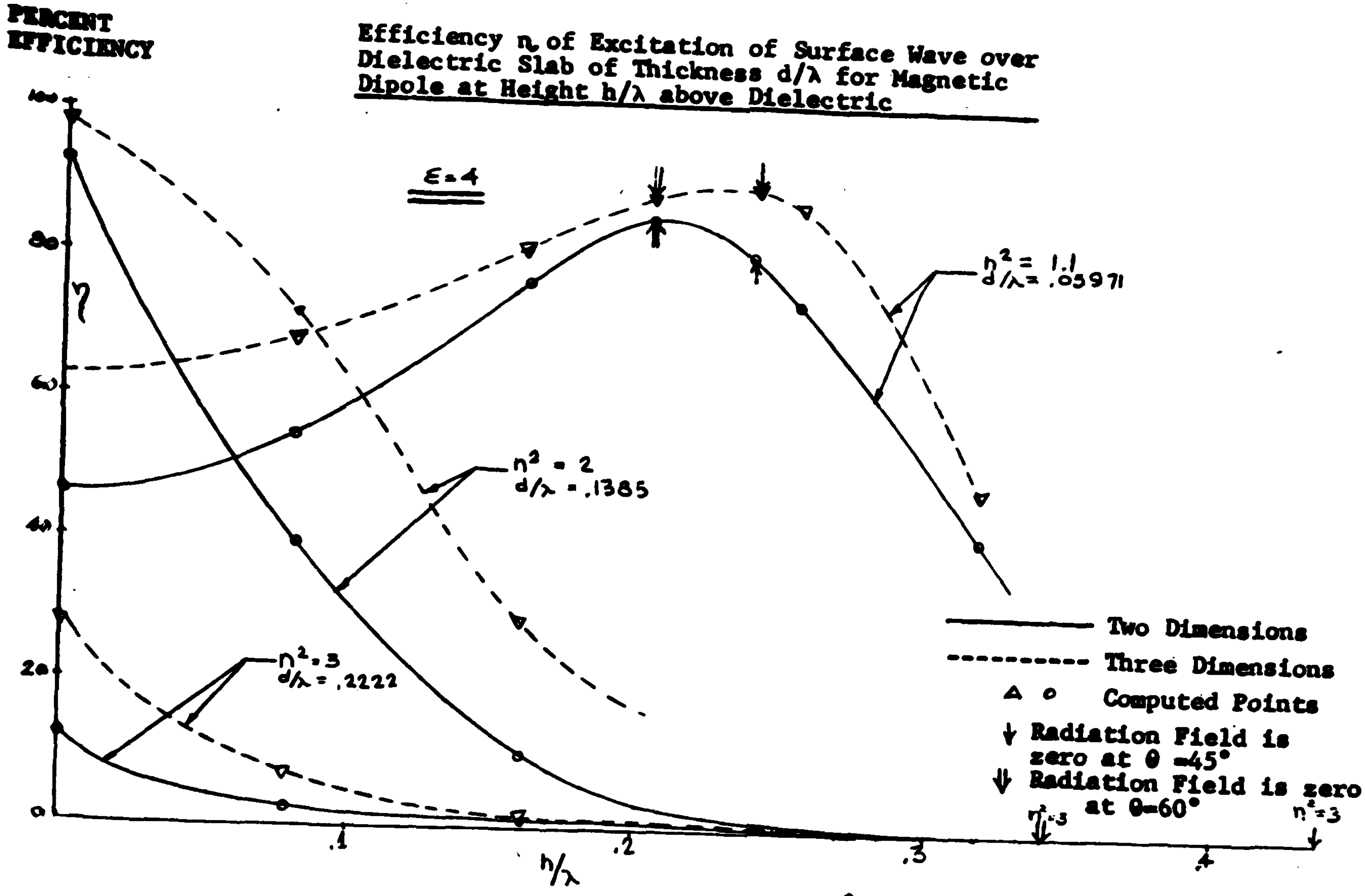


Figure 9b

$n^2 = 2$:
↓ point at .6875; ⇓ point at .5400

Efficiency η of Excitation of Surface Wave over Dielectric Slab of Thickness d/λ for Magnetic Dipole at Height h/λ above Dielectric

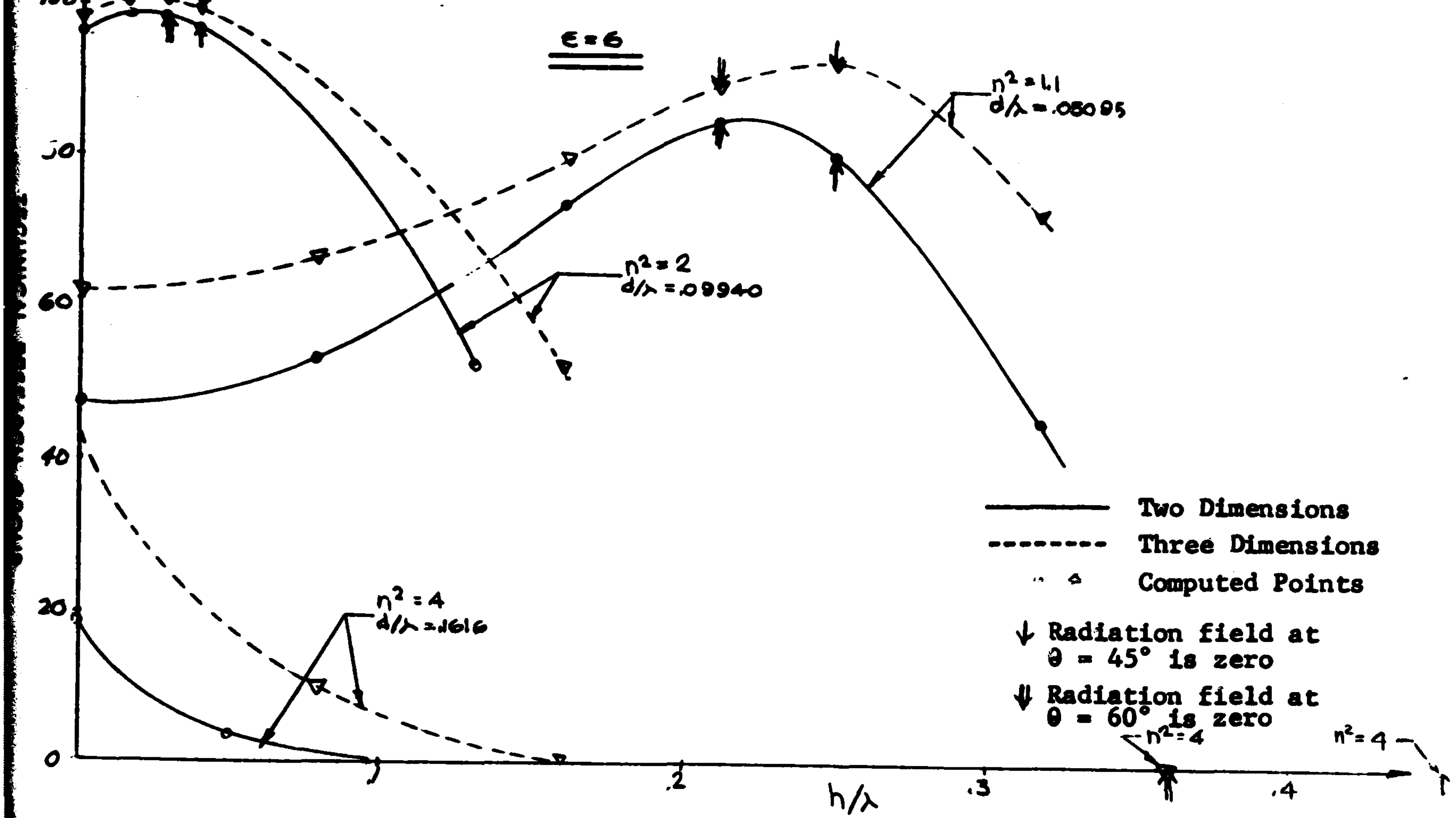


Figure 9c

Efficiency η of Excitation of Surface Wave over Dielectric Slab of Thickness d/λ for Magnetic Dipole at Height h/λ above Dielectric

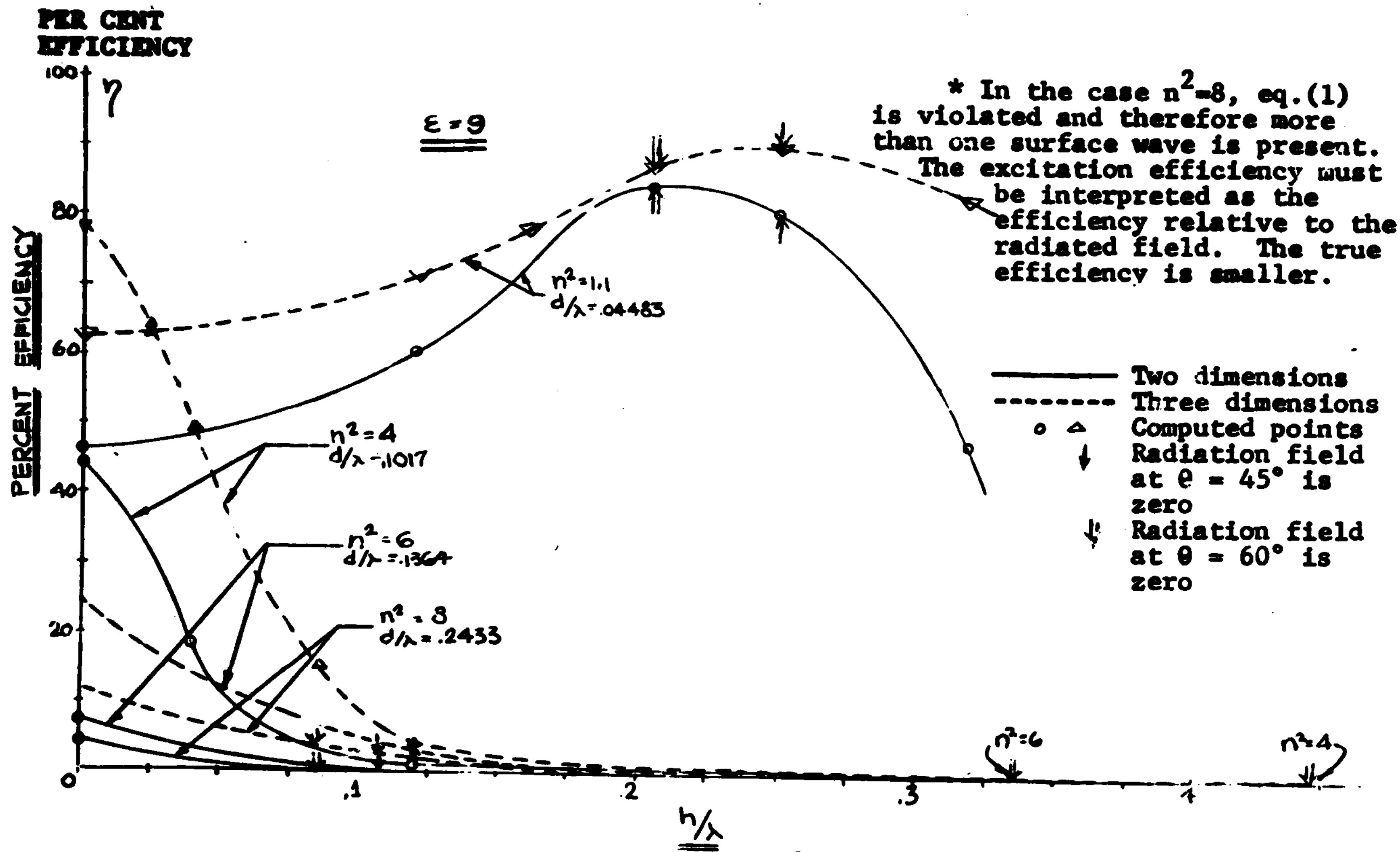


Figure 9d

$n^2 = 6$ ↓ point at h/λ = .4278

$n^2 = 4$ ↓ point at h/λ = .5766

ERRATA SHEET

Page 33, Line 9 and Page 34, Line 8, replace

"$1 + X^2$" by "$\sqrt{1 + X^2}$"

Page 33, Line 7, at end of sentence insert:

"The stationary phase evaluation of the ψ integration reduces this factor to $(1+X^2)^{1/4}$."

Page 43, Line 9 - replace "n" by "$\sqrt{n}$"

Page 43, Line 10 - replace "n^2" by "n"

Equation (17) - delete factor "n" in numerator.

To the references add:

[7] Fernando and Barlow, "An Investigation of the Properties of Radial Cylindrical Surface Waves Launched over a Flat Reactive Surface", Proc. of the Inst. of Elect. Eng., Part B, May 1956, p. 307.

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2 Kay, A. F., "The Excitation of Surface Waves in Multi-layered Media", Technical Research Group, AF19(604)-1126, Oct. 1954 (available upon request)

3 Planck, Max, "Theory of Light", MacMillan Company, London 1932, pp. 121-153

4 Elliott, R. S., "On the Theory of Corrugated Surfaces," Trans. of PGAP, April 1954.

5 Friedman, B., and Williams, W., "Excitation of Surface Waves", RR No. EM-99, New York University, Institute of Math. Sciences, AF19(604)-1717, October 1956.

6 Friedman, B., "Modes in Anisotropic Structures", Technical Research Group, AF19(604)-1015, Jan. 1955